Midterm Exam Abstract Algebra, Spring 2005

Rules: This is an open textbook, open notes, open mind, but not open-mouth, open world-wide-web search, open library search, or open tutoring service. Thus, you can ask anyone anything about any homework problems, but ask no one (except me) anything about these problems. You may quote any results in our text up to Section 11, or any results proven in the homework up to Section 11, but not beyond. Any proofs should include all steps using the identity element, associativity, commutativity, or other relevant group properties.

1. For each of the following sets of conditions, find a binary operation on three elements that satisfy the following conditions, or indicate why it cannot be done. Please place your answers on the worksheet provided. Note there are many thousands of different binary tables, so potentially there could be many answers to a given problem, but you need only give one example.
   (a.) identity, associative, commutative
   (b.) identity, associative, not commutative
   (c.) identity, not associative, commutative
   (d.) identity, not associative, not commutative
   (e.) no identity, associative, commutative
   (f.) no identity, associative, not commutative
   (g.) no identity, not associative, commutative
   (h.) no identity, not associative, not commutative

2. Show that the following is a group (just fyi, it is called the centralizer of the group $G$ since it is the set of elements that commute with every element of $G$):
   \[ C = \{ x \in G \mid gx = xg, \forall g \in G \} \]
   Be careful to quote all relevant results from the text and/or show all steps using all group properties.

3. Let $A : G \to G^0$ be a homomorphism of the group $G$ into the group $G^0$. Let $e^0$ be the identity of $G^0$, and define a subset of $G$ by
   \[ H = \{ x \in G \mid A(x) = e^0 \} \]
   Show that $H$ is a subgroup of $G$. (Just fyi, we will call $H$ the kernel of $A$.)

4. Suppose $G$ is a finite cyclic group of order 88 generated by the element $a$, i.e., $G = \langle a \rangle$. Quoting the appropriate result from the text or homework, explain how you determine the order (and determine the order!) of the following:
   (a.) $a^{21}$
   (b.) $a^{22}$
   (c.) $a^{23}$
   (d.) $a^{24}$
   (e.) Determine how many elements of $G$ are generators.

5. The text discusses the dihedral groups, in particular, the subgroup lattice of $D_4$ is given on pages 79-81. $D_4$ is the group of rigid motions of a square; we now consider $D_6$, the group of rigid motions of a regular hexagon.
   Let $\frac{1}{2} = (1\ 2\ 3\ 4\ 5\ 6)$ in cycle notation; note $\frac{1}{2}$ is the rotation of the hexagon by 60 degrees. Let $1_1 = (2\ 6)(3\ 5), 1_2 = (1\ 3)(4\ 6), 1_3 = (1\ 5)(2\ 4)$ be the flips about diagonals through opposite vertices, and let $2_1 = (1\ 2)(3\ 6)(4\ 5), 2_2 = (1\ 4)(2\ 3)(5\ 6), 2_3 = (1\ 6)(2\ 5)(3\ 4)$ be the flips about diagonals through the midpoints of opposite sides.
   (a.) Find $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{6}$. Identify the even permutations of $D_6$.
   (b.) Write the full Cayley table for this group, using the worksheet provided.
   (c.) Identify the even permutations of $D_6$.
   (d.) Find all subgroups of $D_6$. Just list them; you need not prove they are subgroups.
   (e.) Draw the subgroup lattice for $D_6$.
   (f.) List two elements in $S_6$ which are not in $D_6$. Explain why they are not rigid motions of the hexagon.