Math 5900 Spring 2013 Homework Outline

January 14

Course Overview

Review Math 2600 Euclid's proof of an infinitude of primes

Maple investigation of density of primes

HW:

Use Maple to investigate the number of primes in a random section of integers with x digits, where x is ten times the number of digits in your last name.

Read Guy Section A, submit a short paragraph on two different problems that might interest you.

Jan 16

Euler's proof of the infinitude of primes and the Prime Number Theorem

Handout on Euler's proof, connection with Riemann Hypothesis.

Prime Number Theorem: number of primes up to N is roughly N/log(N), equivalently, the Nth prime p is roughly N log(N).

HW:

Use Maple to investigate the number of primes in a random section of integers with x digits, where x is 10, 20, 30, 40, 50, 60, 70, and determine how close your results are to the predictions of the Prime Number Theorem.

Read Guy Section F, submit a short paragraph on two different problems that might interest you.

Jan 18

Modular (Clock) arithmetic and cyclic groups

It is the stroke of midnight; a falling star stops next to you and says that 1000 hours from now you will meet your true love. What time of the day will you meet your true love?

Fermat's Little Theorem: investigate 2^n mod p for p = 11, 13, 17; what is 2^1000 mod p?

Connections to Math 3500 modern algebra: cyclic groups, subgroups, generator (=primitive root)

Converse of Fermat's Little Theorem: pseudoprimes and primality testing, Carmichael numbers

HW:

Find the powers of 2 modulo 23; of 5 modulo 23; of 2 modulo 29; of 5 modulo 29. What is the period / order of each cyclic subgroup?

Multiply two (five or six digit) primes p and q; define n=pq; use the *nextprime* Maple command to find a prime P that is close to n. Use Maple to calculate 2^{P-1} mod P to verify that Fermat's Little Theorem holds, then calculate 2^{n-1} mod n to see what happens when n is not prime. Repeat this with a new pair p and q.

Verify that 1729 is a Carmichael number, that is, that 2^(1729-1) = 1 mod 1729 but that 1729 is not a prime. (Irrelevant note: 1729 is the famous Hardy-Ramanujan number, also known as the taxicab number; see Wikipedia for its story.)

Read Guy Section E, write up short paragraph on two different problems that might interest you.

Jan 23

Heuristic Arguments

Review Fermat and Mersenne prime conditions from Math 2600 Foundations of Math, in particular, review geometric series.

Heuristic arguments on expected number of Fermat and Mersenne primes based on Prime Number Theorem.

HW:

Give a heuristic argument that there should be an infinite number of primes of the form n^2 + 1. You may assume that the integral int(-1/log(x), x=M..N) diverges to negative infinity as N goes to infinity.

Read Guy Section D, write up short paragraph on two different problems that might interest you.

Jan 25

Euclidean algorithm and gcd

Euclidean algorithm (handout from Euclid's *Elements*), Mersenne primes and perfect numbers.

Calculate gcd(25, 32), have students calculate gcd(1729, 2014) or gcd(1729, 2015) or gcd(1729, 2016)

Solve 25x - 32y = 1, solve 1729x – 2013y = 1.

HW:

Calculate gcd(12345, 543210), solve 43x-52y=1.

Read Guy Section C, write up short paragraph on two different problems that might interest you.

Jan 28

Solving linear equations

Stamp or coin problems: given 17 cent and 39 cent stamps, find how many of each stamp one needs to get exactly $6.66 postage. (Alternate: Given 43 and 52 cent stamps, find how many of each stamp one needs to get exactly $17.29 postage.)

Chinese remainder theorem (do one in class):
(Sun Tsu, 4th century) There are certain things whose number is unknown. Repeatedly divided by 3, the remainder is 2; by 5 the remainder is 3; and by 7 the remainder is 2. What will be the number?
(Brahmagupta, 7th century) An old woman goes to market and a horse steps on her basket and crashes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same happened when she picked them out three, four, five, and six at a time, but when she took them seven at a time they came out even. What is the smallest number of eggs she could have had?
(http://www.cut-the-knot.org/blue/chinese.shtml)

Show no solution: x = 1 mod 15, x = 3 mod 21.

HW:

Solve this stamp problem: given 5 cent and 7 cent stamps, how many of each would you use to get 53 cents in postage?

Use the Chinese Remainder Theorem to find a solution for x, provided a solution exists.
x = 3 mod 5, x = 5 mod 7
Use the Chinese Remainder Theorem to find a solution for x, provided a solution exists.
x = 4 mod 6, x = 5 mod 8

Read Guy Section B, write up short paragraph on two different problems that might interest you.

Jan 30

Solving quadratic equations

Review usual quadratic formula derivation: x^2 + 4x – 1 = 0 mod p. Solve for prime p=11, 13, 19 (using primitive root 2)

Legendre symbol (contrast notation for Legendre symbol with notation for binomial coefficient), connection with even-odd powers of generator of cyclic subgroup mod p.

Calculate Legendre symbol L(5,7), L(5, 11), L(5, 13) via a^{(p-1)/2} mod p (if time, also via Gaussian reciprocity)

HW:

Find the Legendre symbols (refer to a previous homework exercise of Jan 18):

$\left(\frac{3}{23}\right)$, $\left(\frac{5}{23}\right)$, $\left(\frac{15}{23}\right)$, $\left(\frac{3}{29}\right)$, $\left(\frac{5}{29}\right)$, $\left(\frac{15}{29}\right)$.

Complete the square and determine if this equation has a solution: $x^2 + x + 1 = 0 \bmod 29$.
Complete the square and determine if this equation has a solution: $x^2 + x + 1 = 0 \bmod 31$.

Read the article on mathematical writing by Steven Krantz (*A Primer of Mathematical Writing*, Amer. Math. Soc. (1996), pages 14-17, 25- 29, 33-41, posted in Blackboard) and the brief discussion of writing by Dr. Deanin (see Blackboard). Write a short paragraph indicating a few items that you found interesting.

Feb 1

Algebraic number fields

Gaussian integers ($a + b\,i$, norm $a^2 + b^2$, rational primes = -1 mod 4), Eisenstein integers ($a + b\,\omega$, norm $a^2 - ab + b^2$, rational primes = -1 mod 3)

Q(sqrt(-5)), 6=2*3 = (1+sqrt(-5))(1-sqrt(-5)), ideals motivated by gcd, class number and connection to quotient groups in Math 3500 modern algebra.

HW:

Looking at pages 55-56, problem A16 in the Guy text, identify five Eisenstein primes not on the x-axis, either from the definition or from the picture (ideally both ways!)

By now you should have been thinking about a problem: work on it and verify that it is the one you want to commit to.

Feb 4

More notation

Sum of divisors function, Euler phi function, Big O, little o notation, finite fields, p-adic fields, other topics relevant to the particular projects your colleagues have chosen.

HW:

Submit one page writeup which briefly introduces your chosen problem.