Multiplicative Persistence of nonzero fixed point digits

Colin Lubner
Robert Styer
(correspondence: Department of Mathematics and Statistics, Villanova University, 800 Lancaster Avenue, Villanova, PA 19085–1699, phone 610–519–4845, fax 610–519–6928, robert.styer@villanova.edu)

Keywords: persistence, digit problem, multiplicative persistence, iterated digit functions
Mathematical Subject Classification 2010: Primary: 00A08, 97A20

Abstract

If $f(n)$ is the product of the digits of f, we define the multiplicative persistence of n as the number of iterations k needed so $f^k(n)$ is a single digit. Almost all positive integers have a fixed point zero; we investigate those cases whose fixed point is a nonzero digit. In bases 4, 6, 10, and 12, if the fixed point is relatively prime to the base, the maximum persistence is one.

1 Introduction

Neil J. A. Sloane, perhaps best known as the creator and maintainer of the Online Encyclopedia of Integer Sequences (OEIS), defines the multiplicative persistence of a number as the number of steps to reach a one-digit number when repeatedly multiplying the digits together [9]. (As a side note, Sloane is also the author of two rock-climbing guides to New Jersey [6], [7].) He conjectured that the number of iterates needed to reach a fixed point is bounded, in particular, in base 10, he conjectured that one needs at most 11 iterates to reach a single digit.

Definition 1. *Let* $n = \sum_{j=0}^{r} d_j B^j$*, with each* $0 \le d_j < B$*, be the base* B *expansion of* n*. We define the digital product function as* $f(n) = \prod_{j=0}^{r} d_j$*.*

The persistence of a number n *is defined as the minimum number* k *of iterates* $f^k(n) = d$ *needed to reach a single digit* d*. We will say that* n *collapses to the digit* d*.*

It is well known that every number has finite persistence, and that almost all numbers iterate to zero (see [1], [2], [3], [4], [8], [10]). In this paper we consider the iterates that collapse to nonzero digits in base 10.

Example: Let $n = 5579$. Then $f(5579) = 5 \cdot 5 \cdot 7 \cdot 9 = 1575$, then $f(1575) = 1 \cdot 5 \cdot 7 \cdot 5 = 175$, then $f(175) = 1 \cdot 7 \cdot 5 = 35$, then $f(35) = 3 \cdot 5 = 15$, and $f(15) = 1 \cdot 5 = 5$; in other words, $f^5(5579) = 5$, so 5579 has persistence 5.

One easily sees that $n = 9755$ or $n = 137155311$ or $n = 191151117511$ also have persistence 5, since each of these has $f(n) = 1575$. Thus, adding or removing the digit 1 does not change the persistence, nor does rearranging the digits, nor does replacing digits that are products of smaller digits by these smaller digits (such as replacing 9 by two digits 3) affect the persistence.

We claim that the maximum persistence of a number that collapses to 1, 3, 7, or 9 is 1. We also will find probable bounds for the persistence of a number that collapses to 2, 4, 5, 6, or 8.

2 Fixed points relatively prime to the base

Looking at a table of values of n that collapse to a nonzero digit (see [11]), one notes that numbers which collapse to 1, 3, 7, and 9 look quite simple.

The easiest case is when $f(n_1) = 1$. This can only happen when each digit of n_1 is 1, that is, $n_1 = (10^m - 1)/9$ for some positive integer m. Now suppose $f(f(n_2)) = 1$. Then $f(n_2) = (10^m - 1)/9$, so the product of the digits of n_2 must end in a 1 (that is, viewing the product modulo 10), thus n_1 has no even digits nor a digit 5. Thus, the only possible digits of n_2 are 1, 3, 7, and 9, hence $f(n_2) = 3^a 7^b$ for some nonnegative integer powers a and b. Note that $3^{20} \equiv 1 \bmod 100$ and $7^4 \equiv 1 \bmod 100$. One can verify that $3^a 7^b \not\equiv 11 \bmod 100$ for any $0 \le a < 20$ and $0 \le b < 4$. Thus, it is not possible for any n_2 to have $f(n_2)$ be a repunit with all digits 1. In other words, the only numbers that iterate to 1 are repunits $(10^m - 1)/9$.

Amazingly, this same idea works for 3, 7, and 9 as well.

Theorem 1. *Suppose $f^k(n) = 1$, 3, 7, or 9 for some iterate k. Then n has maximum persistence 1. More precisely:*

- *Suppose n collapses to 1. Then $n = (10^m - 1)/9$ for some positive integer m.*
- *Suppose n collapses to 3. Then n has a single digit 3 and the rest of the digits are 1.*
- *Suppose n collapses to 7. Then n has a single digit 7 and the rest of the digits are 1.*
- *Suppose n collapses to 9. Then n has either a single digit 9 or exactly two digits 3, and the rest of the digits are 1.*

Proof. Suppose there is a number n with persistence $k \ge 2$ collapsing to 1, 3, 7, or 9. Let $n_2 = f^{k-2}(n)$ and $n_1 = f^{k-1}(n)$. Since $f(n_1) = 1$, 3, 7, or 9, then every digit of n_1 must be 1, except for a possible single digit of 3 or 7 or 9, or two digits equaling 3. In particular, the last digit of n_1 cannot be even or 5. Since the last digit of n_1 is not even nor 5, n_2 cannot have any even digit or a digit of 5. So the only possible digits of n_2 are 1, 3, 7, or 9. Thus, $f(n_2) = n_1 = 3^a 7^b$ for some nonnegative integers a and b.

We look at $3^a 7^b$ modulo 100. Note 3 is of order 20 modulo 100, and 7 is of order 4 modulo 100. One can verify that for every choice of $0 \le a < 20$ and $0 \le b < 4$, the next to last digit (the tens digit) is always even. Since every n_1 has all odd digits, $3^a 7^b$ can never equal n_1. So it is impossible to find any n_2 with $f(n_2) = n_1$, proving our theorem. □

This type of argument can never work if $B + 1$ is composite; if base B has $B + 1 = d_1 \cdot d_0$, then $f^2(d_1 \cdot B + d_0) = f(d_1 \cdot d_0) = f(B + 1) = 1$. If $B + 1$ is prime, this type of argument still might fail, for instance, in base 16, $f^2(37D) = f(111) = 1$, while in base 18, $f^2(777) = f(111) = 1$. But the same base 10 argument does work for bases 4, 6, and 12.

Theorem 2. *Let the base $B = 4$. Suppose $f^k(n) = 1$ or 3. Then n has maximum persistence 1.*

Let the base $B = 6$. Suppose $f^k(n) = 1$ or 5. Then n has maximum persistence 1.

Let the base $B = 12$. Suppose $f^k(n) = 1$ or 5 or 7 or the digit 11. Then n has maximum persistence 1.

Proof. Consider base $B = 4$. Suppose there is a number n with persistence $k \ge 2$ collapsing to 1 or 3. Let $n_2 = f^{k-2}(n)$ and $n_1 = f^{k-1}(n)$. Since $f(n_1) = 1$ or 3, then every digit of n_1 must be 1, except for a possible single digit of 3. In particular, the last digit of n_1 must be 1 or 3. Since the last digit of n_1 is not 0 or 2, n_2 cannot have a 0 or 2 digit. So the only possible digits of n_2 are 1 and 3, so $f(n_2) = 3^a$ for some positive integer a. Note that $3^4 \equiv 1 \bmod 4^2$. Since, in base 4, $3^1 \equiv 03 \bmod 100$, $3^2 \equiv 21 \bmod 100$, $3^3 \equiv 23 \bmod 100$ and $3^4 \equiv 01 \bmod 100$, we see that the second digit (the 'fours' digit) is always even. But every n_1 has all odd digits, so 3^a can never equal n_1. Thus, it is impossible to find any n_2 with $f(n_2) = n_1$, proving the base 4 case.

Similarly, consider base $B = 6$. Suppose there is a number n with persistence $k \geq 2$ collapsing to 1 or 5. Let $n_2 = f^{k-2}(n)$ and $n_1 = f^{k-1}(n)$. Since $f(n_1) = 1$ or 5, then every digit of n_1 must be 1, except for a possible single digit of 5. In particular, the last digit of n_1 must be 1 or 5. Since the last digit of n_1 is not 0, 2, 3, or 4, n_2 cannot have a 0, 2, 3, or 4 digit. So the only possible digits of n_2 are 1 and 5, so $f(n_2) = 5^a$ for some positive integer a. Note that $5^6 \equiv 1 \bmod 6^2$. In base 6, $5^1 \equiv 05 \bmod 100$, $5^2 \equiv 41 \bmod 100$, $5^3 \equiv 25 \bmod 100$, $5^4 \equiv 21 \bmod 100$, $5^5 \equiv 45 \bmod 100$ and $5^6 \equiv 01 \bmod 100$; we see that the second digit (the 'sixes' digit) is always even. Since every n_1 has only digits 1 and 5, 5^a can never equal n_1. So it is impossible to find any n_2 with $f(n_2) = n_1$, proving the base 6 case.

Finally, consider base $B = 12$. Suppose there is a number n with persistence $k \geq 2$ collapsing to 1, 5, 7, or the digit '11'. Let $n_2 = f^{k-2}(n)$ and $n_1 = f^{k-1}(n)$. Since $f(n_1) = 1$, 5, 7, or the digit '11', then every digit of n_1 must be 1, except for a possible single digit of 5 or 7 or '11'. In particular, the last digit of n_1 must be 1, 5, 7, or '11'. Since the last digit of n_1 is not even or divisible by 3, n_2 cannot have digit which is even or divisible by 3. So the only possible digits of n_2 are 1, 5, 7, or '11',so $f(n_2) = 5^a 7^b 11^c$ for some nonnegative integers a, b, c (at least one exponent is positive). Since 5,7, and 11 have orders 12, 6, and 12 modulo 12^2, respectively, we only need to check $5^a 7^b 11^c$ modulo 12^2 for $0 \leq a < 12$, $0 \leq b < 6$, and $0 \leq c < 12$. One can verify that each $5^a 7^b 11^c$ has an even digit in the second place (the "tens" digit in duodecimal). Since every n_1 has only odd digits, $5^a 7^b 11^c$ can never equal n_1. So it is impossible to find any n_2 with $f(n_2) = n_1$, proving the base 12 case of the theorem. □

3 Bounds on persistence of numbers collapsing to 2, 4, 5, 6, 8

We now return to base $B = 10$. In the previous section we dealt with the cases when $f^k(n) = 1$, 3, 7 or 9. We now find lower bounds on the persistence of the other nonzero decimal digits.

Theorem 3. *Let $n < 10^{1000}$ be a positive integer.*

If $f^k(n) = 2$, the persistence k is at most 6; only n with $f(n) = 2^6 7^8$ have persistence 6.

If $f^k(n) = 4$, the persistence k is at most 5; only n with $f(n) = 2^{23} 3^7 7^1$ have persistence 5.

If $f^k(n) = 5$, the persistence k is at most 5; only n with $f(n) = 3^5 5^1 7^2$ or $3^2 5^2 7^1$ or $3^2 5^2 7^3$ have persistence 5.

If $f^k(n) = 6$, the persistence k is at most 8; only n with $f(n) = 2^6 3^{27} 7^1$ have persistence 8.

If $f^k(n) = 8$, the persistence k is at most 6; only n with $f(n) = 2^{20} 3^5 7^1$ or $2^{39} 3^3 7^2$ have persistence 6.

Proof. As before, we write $f(n)$ in terms of the digits $f(n) = 1^a \cdot 2^b \cdot 3^c \cdot 4^d \cdot 5^e \cdot 6^f \cdot 7^g \cdot 8^h \cdot 9^i$ for some nonnegative integers a, b, c, d, e, f, g, h, and i. We replace each 4 with two 2s, each 6 with a 3 and a 2, each 8 with three 2s, and each 9 with two 3s. Thus, $f(n) = 2^\alpha \cdot 3^\beta \cdot 5^e \cdot 7^g$ where $\alpha = b + 2d + 3h$ and $\beta = c + 2i$. If $\min(\alpha, e) > 0$ then the last digit of $f(n)$ would be zero, so n would collapse to zero, a case we are not considering in this paper. Thus, we only need to consider $f(n) = 2^\alpha \cdot 3^\beta \cdot 7^g$ or $f(n) = 3^\beta \cdot 5^e \cdot 7^g$. Since we are assuming n has at most 1000 digits, we have $\lceil \alpha/3 \rceil + \lceil \beta/2 \rceil + g \leq 1000$ and $\lceil \beta/2 \rceil + e + g \leq 1000$. Using Maple, we check the persistence for each $f(n) = 2^\alpha \cdot 3^\beta \cdot 7^g$ or $f(n) = 3^\beta \cdot 5^e \cdot 7^g$ that collapse to either 2, 4, 5, 6, or 8, with the restrictions $\lceil \alpha/3 \rceil + \lceil \beta/2 \rceil + g \leq 1000$ and $\lceil \beta/2 \rceil + e + g \leq 1000$. We find that none have persistence greater than the listed maxima for each case, and that the maxima in this range are reached only by the listed cases. □

Based on similar calculations,we have these conjectural bounds on persistence: in base 4, the maximum persistence of any number collapsing to 2 is 3; in base 6, the maximum persistence of a number collapsing to the digit 2 or digit 4 is 5, to the digit 3 is 2; in base 12, the maximum persistence of a number collapsing to digits 2 or 10 is 3, to digit 3 is 4, to digit 4 is 6, to digits 6 or 8 is 5, to digit 9 is 4.

References

[1] Beeler, M., Gosper, R.W., and Schroeppel, R. HAKMEM. MIT AI Memo 239, Feb. 29, 1972 `http://www.inwap.com/pdp10/hbaker/hakmem/number.html#item56`

[2] Carmody, P., OEIS A003001, and a 'Zero-Length Message'. 23 Jul 2001. `http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0107&L=NMBRTHRY&P=R1036&I=-3.`

[3] M. R. Diamond, Multiplicative persistence base 10: some new null results, *JoOUP* **3** (2010), 1–10 `http://www.markdiamond.com.au/download/joous-3-1-1.pdf.`

[4] Gottlieb, A. J., Problems 28-29 in "Bridge, Group Theory, and a Jigsaw Puzzle." *Technology Rev.* **72**, Dec 1969, unpaginated

[5] Guy, Richard, "F25", *Unsolved Problems in Number Theory* 3rd edition, Springer 2004.

[6] Sloane, Neil and Nick, Paul, *Classic Rock Climbs No. 5: New Jersey Crags, New Jersey.* Chockstone Pr: 1999

[7] Nick, Paul and Sloane, Neil. *Rock Climbing New Jersey.* Falcon Publishing, Guilford: 2000. Print.

[8] Perez, Stephanie and Styer, Robert. Persistence: A Digit Problem. **Involve: A Journal of Mathematics 8**, 3 (2015), 439–446.

[9] Sloane, Neil J. A., The Persistence of a Number. *Journal of Recreational Mathematics* **6**, 2 (1973), 97–98.

[10] Wagstaff, S. W., Iterating the Product of Shifted Digits, *Fibonacci Quarterly* **19** (1981), 340–347.

[11] Weisstein, Eric W., "Multiplicative Persistence." From MathWorld–A Wolfram Web Resourc http://mathworld.wolfram.com/MultiplicativePersistence.html, accessed 26 June 2012.