revised June 15, 2006

Title: Theorem 6 when a is composite

In this note we treat the equation

$$\pm a^x \pm b^y = \pm a^w \pm b^z \tag{1}$$

for relatively prime positive integers $a > b > 1$ such that the largest prime divisor of a is not a base-b Wieferich prime. We are concerned only with solutions (x, y, w, z) such that the pair (x, y) is distinct from the pair (w, z). In what follows, we will always be dealing with (1) under these restrictions.

Theorem 1 If the greatest of the four terms in (1) is a power of 2, then (1) must be one of the equations in (A) or (B) below.

$$\begin{aligned}
(A): 2^3 - 3 &= 2^5 - 3^3 \\
2^4 - 3 &= 2^8 - 3^5 \\
2^3 - 5 &= 2^7 - 5^3 \\
3 + 2 &= 2^5 - 3^3 \\
3^2 + 2^2 &= 2^8 - 3^5 \\
5 + 2 &= 2^5 - 5^2 \\
3^2 - 2^2 &= 2^5 - 3^3 \\
5 - 2 &= 2^7 - 5^3 \\
11 - 2^2 &= 2^7 - 11^2 \\
(B): M + 2 &= 2^{u+1} - M \\
M + 2^u &= 2^{2u} - M^2 \\
F - 2 &= 2^{v+1} - F
\end{aligned}$$

where $M = 2^u - 1$ and $F = 2^v + 1$.

Proof: Letting $x_1 = \min\{x, w\}$, $y_1 = \min\{y, z\}$, $x_2 = \max\{x, w\}$, $y_2 = \max\{y, z\}$, and $b = 2$, we rewrite (1) as

$$2^{y_2} - a^{x_2} = (-1)^g 2^{y_1} + (-1)^h a^{x_1} \tag{2}$$

where g and h are in the set $\{0, 1\}$. We cannot have $g = h = 1$. If $g = 0$ and $h = 1$, then, by Theorem 4 of [Sc], (1) must be one of the first three equations listed in (A) above. So we can assume $h = 0$ and write (2) as

$$2^{y_1}(2^{y_2-y_1} - (-1)^g) = a^{x_1}(a^{x_2-x_1} + 1). \tag{3}$$

Let m be defined as in Lemma 1 of [Sc-St2] for $b = 2$. By Lemma 9 of [Sc-St2], we must have $x_1 = m$. Let v be the least number such that $a | 2^v - (-1)^g$. Let the prime factorization of a be $a = \prod_i {p_i}^{\alpha_i} \prod_j {q_j}^{\beta_j}$, where for each i, ${p_i}^{\alpha_i} || 2^v - (-1)^g$, and for each j, ${q_j}^{\beta_j + 1} | 2^v - (-1)^g$. From (3) we see that we must have

$$v \prod_i {p_i}^{(m-1)\alpha_i} | y_2 - y_1 \tag{4}$$

and, for any q_j which is not a Wieferich prime,

$$q_j | v \tag{5}$$

and

$$q_j | r - 1 \tag{6}$$

where r is some prime divisor of a.

Now assume $m > 1$. Then, noting that $a \neq 2^n + 1$ for any integer n, we see that

$$2^v > a. \tag{7}$$

From (6) we see that, since $m > 1$, the largest prime divisor of a must be greater than or equal to 7. Also, by the restriction on a in (1), the largest prime divisor of a must be among the p_i. Combining this with (4) and (7), we obtain

$$k \frac{\log a}{\log 2} 7^{m-1} = y_2 \tag{8}$$

where $k \geq 1$.

Now let $\Lambda = y_2 \log 2 - x_2 \log a$, let $G = y_2 / \log a$, and let $c = a^{x_1} + (-1)^g 2^{y_1} = a^m + (-1)^g 2^{y_1}$. From (3) we see that $1 < c < 2a^m$. Using a theorem of Mignotte [Mi] with parameters chosen by Bennett [Be, Section 6], we see that we must have either

$$G < 2409.08 \tag{9}$$

or

$$\log \Lambda > -22.997 (\log G + 2.405)^2 \log a \log 2. \tag{10}$$

In exactly the same manner as in the proof of Theorem 2 in [Sc-St2], we obtain

$$G < 2 \frac{\log c}{\log a \log 2} + 22.997 (\log G + 2.405)^2. \tag{11}$$

(Note that (11) corresponds to (11) in [Sc-St2].) Using (8), along with the definition of G and c, we find that we can view (11) as an inequality in the variables k and m. If (11) holds for $m \geq 6$ and $k \geq 1$ then it must hold for $m = 6$ and $k = 1$. But then both (11) and (9) require $7^5 < 2409.08 \log 2$, false. So we can assume $m < 6$ in which case both (11) and (9) require $G < 2409.08$, so that

$$x_2 / \log 2 < y_2 / \log a < 2409.08$$

giving $x_2 \le 1669$. From this we get

$$2^{y_2} = a^{x_2} + c \le a^{1670}. \tag{12}$$

Let p be the largest prime dividing a, recalling that p must be among the p_i. Combining (4), (12), and (7), we obtain

$$p < 1670. \tag{13}$$

Assume that none of the primes q_j is a Wieferich prime. Then combining (4) and (5), we see that $y_2 > a$, so that (12) gives $a < 24333$. If $p > 547$, then we can use (4) and (12) to obtain

$$v < \frac{1670}{p} \frac{\log a}{\log 2} \le 43. \tag{14}$$

There are only nine primes p in the range $547 < p < 1669$ which allow $v \le 43$, and these can easily be shown to make (2) impossible since each requires the left side of (3) to be divisible by primes which cannot divide the right side of (3). Now we can use a computer search to show there are no cases of (2) with $m > 1$ and with none of the q_j Wieferich: we use $a < 24333$, $p < 547$, and $vp|y_2 - y_1$, and use the method called "bootstrapping" in [Sc-St1, page 217] to find primes which must divide the left side of (3) but cannot divide the right side of (3).

Now suppose at least one of the q_j is Wieferich. By (13), the only such Wieferich prime possible is 1093, and $1093||a$. Combining (4) and (5), we see that $y_2 > a/1093$ so that

$$\frac{a}{1093} \log 2 < 670 \log a$$

from which we obtain $a < 46491482$. Since $p \ge 1097$, we can use (4) and (12) to obtain $v < 39$, impossible when $1093|a$.

Thus we have $x_1 = m = 1$, and we can use the methods of the proof of Theorem 6 of [Sc-St2] to complete the proof.

Comment: The theorem we have just proven generalizes Theorem 6 of [Sc-St2] for the Type 3 and Type 4 cases. The Type 2 case can be handled just as easily the same way, requiring additional calculations which we have not yet done but which are just as practical as those in the above proof. The Type 1 case can also be handled in the same way, but here we must use the bounds obtained on a and x_2 to show that the theorem holds except for a finite number of computable exceptions, the numbers involved being too large to allow a practical computer search. Indeed, if we do not require the calculations to be practical, we can obtain the following, letting $x_1 = \min\{x, w\}$:

Theorem 2

Let B and W be fixed positive real numbers. Then, if (1) holds with $b < B$ and the highest prime dividing a is not a base-b Wieferich prime greater than W, we must have $x_1 = m = 1$, except for a finite number of effectively computable cases.

[Sc-St2] R. Scott and R. Styer, On the generalized Pillai equation $\pm a^x \pm b^y = c$, *Journal of Number Theory*, **118** (2006), pp. 236–265.