Dear Gerry,

I am sorry I cannot attend the West Coast Number Theory Conference this year. Reese Scott had an interesting question that we hope you might be able to add to the list of annual WCNT questions this year. Hope the conference goes well.

Sincerely, Bob Styer and Reese Scott

When are there two solutions (x, y, u, v) to the equation $(-1)^u * a^x + (-1)^v * b^y = c$ for given integers $a, b > 1$ and integer $c > 0$?

Here is what is already known
(see Michael Bennett, On some exponential equations of S. S. Pillai, {\it Canadian Journal of Mathematics}, {\bf 53}, no. 5, (2001), 897--922;
Reese Scott and Robert Styer, On the generalized Pillai equation $\pm a^x \pm b^y = c$, {\it Journal of Number Theory}, {\bf 118} (2006), pp. 236--265.)

All cases with three or more solutions are known. In these cases (a,b,c) is one of the following (taking $a>b$):
(3,2,1)
(3,2,5)
(3,2,7)
(3,2,11)
(3,2,13)
(4,3,13)
(5,2,3)

This leaves the question when there are exactly two solutions (x_1, y_1, u_1, v_1) and (x_2, y_2, u_2, v_2). Noting that x and y uniquely determine u and v, we write $x_1 < x_2$ and $y_1 < y_2$.

There are three known infinite families of (a,b,c) giving exactly two solutions:
(1.) $x_1 = x_2$
(2.) $| a^m - b^n | = 1$ with $x_2 = 2 x_1$ and $y_2 = 2 y_1$
(3.) $(a,b,c) = (4,2,2^{2k+1})$.
In addition, there are two trivial infinite families:
(1.) $a^{x_1} = b^{y_2}$, $a^{x_2} = b^{y_1}$
(2.) $a=b=2$ and $x_i = y_j$ for some $1 \le i,j \le 2$.
All known cases of exactly two solutions to the equation other than ones derived from the known cases with three or more solutions and the five known infinite families mentioned above can be derived from the following fourteen anomalous cases:

$13 - 3 = 13^3 - 3^7 = 10$
$91 - 2 = 91^2 - 2^{13} = 89$
$2 + 5 = 2^5 - 5^2 = 7$
$15 - 6 = 15^2 - 6^3 = 9$
$5 + 279 = 5^7 - 279^2 = 284$
$4930 - 30 = 4930^2 - 30^5 = 4900$
$6^4 - 3^4 = 6^5 - 3^8 = 1215$
$2^2 + 6^2 = 2^8 - 6^3 = 40$
$15^2 + 5^2 = 15^3 - 5^5 = 250$
$11 - 2^2 = 2^7 - 11^2 = 7$
$40^2 - 2^6 = 2^{16} - 40^3 = 1536$
$5 - 3 = 3^3 - 5^2 = 2$

$6^2 - 3^2 = 3^5 - 6^3 = 27$
$21^2 - 98 = 98^2 - 21^3 = 343$

By "derived from" we mean:
(1.) (a,b,c) is considered the same as (b,a,c)
(2.) we disregard duplications due to a or b being a perfect power
(3.) we disregard rearrangements of terms
(4.) and we note that, from any (a,b,c) for which the equation has two solutions (x_1, y_1) and (x_2, y_2) with $x_2= 2 x_1$ we can derive a new (a,b,c) using
$a^{ 2 x_1 } \pm a{x_1} = (a^{x_1} \pm 1) ^2 \mp (a^{x_1} \pm 1)$
where \pm means plus/minus and \mp means minus/plus.