Comments on the equation $\pm ra^x \pm sb^y = c$.

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For given integers $a > 1$, $b > 1$, $c > 0$, $r > 0$, and $s > 0$, we consider N, the number of solutions (x, y, u, v) to the generalized Pillai equation

$$(-1)^u ra^x + (-1)^v sb^y = c \tag{P}$$

in nonnegative integers x, y and integers u, $v \in \{0, 1\}$. Note that the choice of x and y uniquely determines the choice of u and v, so we will usually refer to a solution (x, y).

The Case $(ra, sb) = 1$

There are only a finite number of cases with $N > 3$ solutions to Equation (P) [14]. There are at least five infinite families of cases with $N = 3$ solutions to (P), as well as a number of anomalous cases with $N = 3$ (by 'anomalous case' we mean a case not a member of a known infinite family). Some of these anomalous cases are quite high, e.g., $(a, b, c, r, s) = (56744, 1477, 83810889, 1478, 56743)$, [14]. We have not been able to give a complete finite list of such anomalous solutions, so the question arises: what additional restrictions on the variables would make possible a proof which gives a complete list of anomalous solutions, thus improving the result to $N = 2$ except for completely designated exceptions? This question has been essentially answered with the additional restriction $x > 0$ and $y > 0$ (see [13], in which the problem is reduced to a finite search). But even if only one of the exponents $(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N)$ is equal to zero, the problem becomes more difficult: even with the further restriction $rs = 1$ the methods of [1] and [12] do not suffice without additional heavy restrictions such as placing an upper bound on one of b, c, or $\min(x_1, x_2, \ldots, x_N)$ (when $\min(y_1, y_2, \ldots, y_N) = 0$). But if one adds the yet further restriction that a and b be prime, it is possible to give a complete list of infinite families and a complete list of anomalous solutions, thus obtaining $N = 2$ with completely designated exceptions (see [17]). The restriction that a and b be prime is perhaps not as artificial as it may seem: computer searches in [1] and [12] (supplemented with calculations on the second author's website) suggest that the list of exceptions in Theorem 3 of [17] would remain unchanged even if p and q were allowed to be any relatively prime integers (here of course we would be redefining F and M to allow composite Fermat and Mersenne numbers).

The General Case

In what follows we will refer to a *set of solutions* to (P) which we will write as

$$(a, b, c, r, s : x_1, y_1; x_2, y_2; \ldots; x_N, y_N)$$

and by which we mean the (unordered) set of ordered pairs $\{(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\}$ where each pair (x_i, y_i) gives a solution (x, y) to (P) for given integers a, b, c, r, and s. We say that two sets of solutions $(a, b, c, r, s : x_1, y_1; x_2, y_2; \ldots; x_N, y_N)$ and $(A, B, C, R, S : X_1, Y_1; X_2, Y_2; \ldots; X_N, Y_N)$ belong to the same *family* if a and A are both powers of the same integer, b and B are both powers of the same integer, there exists a positive rational number k such that $kc = C$, and for every i there exists a j such that $kra^{x_i} = RA^{X_j}$ and $ksb^{y_i} = SB^{Y_j}$, $1 \le i, j \le N$.

If $(ra, sb) = 1$, then $k = 1$ and there are only a finite number of sets of solutions in each family; therefore, when $(ra, sb) = 1$, we often dispense with the notion of family and deal simply with sets of solutions.

Equation (P) has been treated by many authors, usually under at least one of the following additional restrictions:

(A.) $\min(x, y) \geq 1$,
(B.) $\min(x, y) \geq 2$,
(C.) $(u, v) = (0, 1)$,
(D.) $(u, v) \neq (0, 0)$
(E.) $\gcd(ra, sb) = 1$,
(F.) $r = s = 1$,
(G.) a is prime,
(H.) a and b are distinct primes,
(I.) a and b are both greater than a fixed real number,
(J.) terms on the left side of (P) are large relative to c

For any combination of such restrictions, we consider the problem of finding a number N_0 such that there are an infinite number of (families of) sets of solutions for which $N = n$ for every n less than or equal to N_0 but only a finite number of (families of) sets of solutions for which $N > N_0$. We also consider the problem of finding a number M such that no sets of solutions have $N > M$, while sets of solutions exist with $N = M$. The following table summarizes some known results, giving, for a given set of restrictions, results on N_0 and M along with citations of sources. In the column headed "Restrictions" we use the letters given in the list above, also writing "K" to mean "no restrictions except those given in (P)."

Restrictions	Results	Sources
A C J	$M \leq 9$	[18]
B C E I	$M \leq 3$	[4]
B C E F I	$M \leq 2$	[4]
A C F G	$M = 2$	[10]
A D F H	$N_0 \leq 2$, $M = 3$	[10]
A F H	$N_0 \leq 2$, $M = 4$	[11]
A C F	$M = 2$	[1]
A F	$N_0 \leq 2$, $M = 4$	[12]
B C E I	$M \leq 2$	[3]
A C E	$M \leq 3$	[3]
C	$N_0 = 3$, $M = 3$	[13]
A E	$N_0 = 2$, $M \geq 4$	[13]
K	$N_0 = 3$, $M = 5$	[14], [15]

Lower bounds on linear forms in logarithms are used for the proofs of all the results cited in the table above except for those in [10] and [11], which are strictly elementary. In [17], we show that strictly elementary methods suffice to improve the results in [10] by eliminating the restriction (A.) and by obtaining a definite value for N_0 in the case with the set of restrictions C, F, G. We can also eliminate the restriction (A.) in improving the result from [11], although here our methods are not strictly elementary (see [17]).

Yet stronger restrictions can give $N_0 = 1$: see for example [8], [1, Theorems 1.3, 1.4, 1.5, and Proposition 2.1], [11, Theorems 2 and 6], [12, Theorems 6 and 7], and [19, Theorem 3]. See also [6] and [2].

Recently, under the restriction CF, we have removed the restriction that a and b be integers and also the restriction that x and y be nonnegative: we find $N_0 = 2$ and $M = 3$ when a and b can be any rational numbers other than 1, and x and y can be any integers, positive, negative, or zero [16].

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