On a conjecture concerning the number of solutions to $a^x + b^y = c^z$, II
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Abstract

Let a, b, c be distinct primes with $a < b$. Let $S(a, b, c)$ denote the number of positive integer solutions (x, y, z) of the equation $a^x + b^y = c^z$. In a previous paper [16] it was shown that if (a, b, c) is a triple of distinct primes for which $S(a, b, c) > 1$ and (a, b, c) is not one of the six known such triples then (a, b, c) must be one of three cases, each of which specifies b and c modulo 24 (with $a = 2$). In the present paper, we eliminate two of these cases (using the special properties of certain continued fractions for one of these cases, and using a result of Dirichlet on quartic residues for the other). Then we show that the single remaining case requires severe restrictions, including the following: $a = 2$, $b \equiv 1 \bmod 48$, $c \equiv 17 \bmod 48$, $b > 10^9$, $c > 10^{18}$; at least one of the multiplicative orders $u_c(b)$ or $u_b(c)$ must be odd (where $u_p(n)$ is the least integer t such that $n^t \equiv 1 \bmod p$); 2 must be an octic residue modulo c except for one specific case; $2 \mid v_2(b-1) \le v_2(c-1)$ (where $v_2(n)$ satisfies $2^{v_2(n)} \parallel n$); there must be exactly two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) with $1 = z_1 < z_2$ and either $x_1 \ge 28$ or $x_2 \ge 88$. These results support a conjecture put forward in [27] and improve results in [16].

1 Introduction

Let $\mathbb{P}$ be the set of positive rational prime numbers. We consider $S(a, b, c)$, the number of solutions in positive integers (x, y, z) to the equation

$$a^x + b^y = c^z, a, b, c \in \mathbb{P}, a < b. \tag{1.1}$$

This paper will continue the discussion of a conjecture on (1.1) found in [16, Conjecture 1.7]:

Conjecture. *For a, b, and c distinct primes with $a < b$, we have $S(a, b, c) \le 1$, except for*
    (i) $S(2, 3, 5) = 2$, $(x, y, z) = (1, 1, 1)$ and $(4, 2, 2)$.
    (ii) $S(2, 3, 11) = 2$, $(x, y, z) = (1, 2, 1)$ and $(3, 1, 1)$.
    (iii) $S(2, 5, 3) = 2$, $(x, y, z) = (1, 2, 3)$ and $(2, 1, 2)$.
    (iv) $S(2, 7, 3) = 2$, $(x, y, z) = (1, 1, 2)$ and $(5, 2, 4)$.
    (v) $S(3, 5, 2) = 3$, $(x, y, z) = (1, 1, 3)$, $(1, 3, 7)$, and $(3, 1, 5)$.
    (vi) $S(3, 13, 2) = 2$, $(x, y, z) = (1, 1, 4)$ and $(5, 1, 8)$.

There is much previous work on various types of exponential Diophantine equations with prime bases (see, for example, [1], [3], [6], [10], [11], [20], [22], [24], [25], [29], [30]). Most such work deals with the familiar Pillai equation $c^z - b^y = a$, taking c and b prime. In 1985 the first author [14] obtained some early results on (1.1) and conjectured that (1.1) has at most one solution in positive integers (x, y, z) with $\min(x, y, z) > 1$. This conjecture is restated in [15] and proven in the introduction to [26]; it is also included in Theorem 1.1 below.

In [25] it is shown that the more general equation

$$(-1)^u p^x + (-1)^v q^y = r^z, p, q, r \in \mathbb{P}, (p, q, r) \neq (2, 2, 2), x, y, z, \in \mathbb{Z}^+, u, v \in \{0, 1\}$$

has at most two solutions (x, y, z, u, v) except when (p, q, r) is a permutation of one of the following: $(5, 3, 2)$ which has seven solutions, $(7, 3, 2)$ which has four solutions, $(11, 3, 2)$ which has three solutions, $(13, 3, 2)$ which has three solutions. But improving this to at most one solution (with listed exceptions) has not been accomplished, even when (u, v) is fixed at $(0, 0)$.

More recently, the following theorem was shown in [16, Lemma 1.2, Theorem 1.4, Theorem 1.5, Corollary 1.6, and Theorem 1.8]:

Theorem 1.1. *If* $(a, b, c) = (2, 3, 5)$, $(2, 3, 11)$, $(2, 5, 3)$, $(2, 7, 3)$, $(3, 5, 2)$ *or* $(3, 13, 2)$, *Equation (1.1) has exactly two solutions. Except for these six cases, if Equation (1.1) has more than one solution, we must have* $(a, b, c) = (2, p, q)$ *for some odd primes* $p > 10^9$ *and* $q > 10^{18}$, *and there must be exactly two solutions* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *as follows:*

$$2^{x_1} + p^{y_1} = q, 2 \mid x_1, 2 \mid y_1, \tag{1.2}$$

and

$$2^{x_2} + p^{y_2} = q^{z_2}, 2 \mid x_2, 2 \nmid y_2, 2 \nmid z_2 > 1. \tag{1.3}$$

Using a result of Bennett [4, Theorem 1.1], and a result of Bauer and Bennett [2, Corollary 1.7] in combination with Theorem 1.1 above, it was shown in [16] that $p \not\equiv 2 \bmod 3$, leading to the following theorem (Theorem 1.5 of [16]):

Theorem 1.2. *If (1.1) has more than one solution and is not one of the six exceptional cases of Theorem 1.1, then we must have* $(a, b, c) = (2, p, q)$ *for some odd primes* p *and* q *satisfying one of the following conditions:*
  $p \equiv 13 \bmod 24$, $q \equiv 5 \bmod 24$,
  $p \equiv 13 \bmod 24$, $q \equiv 17 \bmod 24$,
  $p \equiv 1 \bmod 24$, $q \equiv 17 \bmod 24$.

The purpose of this paper is to first eliminate two of the cases in Theorem 1.2 and then show that the single remaining case implies severe restrictions on p and q. We use the following notation:
  If $2^t \parallel n$, we write $v_2(n) = t$.
  If t is the least positive integer such that $n^t \equiv 1 \bmod p$ for some prime p, we write $u_p(n) = t$.
  We prove the following improvement on Theorem 1.2:

Theorem 1.3. *If (1.1) has more than one solution and is not one of the six exceptional cases of Theorem 1.1, then we must have $(a, b, c) = (2, p, q)$ for some odd primes p and q satisfying all of the following conditions:*

(i): $p \equiv 1 \bmod 48$, $q \equiv 17 \bmod 48$, $2 \mid v_2(p-1) \leq v_2(q-1)$.

(ii): At least one of the multiplicative orders $u_p(q)$ and $u_q(p)$ must be odd.

(iii): 2 is an octic residue modulo q, that is, 2 is congruent to an eighth power modulo q, except when $v_2(p-1) = v_2(q-1) = 4$.

A further restriction is given by the following:

Theorem 1.4. *In Equations (1.2) and (1.3), the following must hold:*

(i): Either $x_1 \geq 28$ or $x_2 \geq 88$.

(ii): If $27 \geq v_2(p-1) = v_2(q-1)$, then $x_2 \geq 88$; and if $87 \geq v_2(p-1)$ and $v_2(p-1) < v_2(q-1)$, then $x_1 \geq 28$.

Theorems 1.3 and 1.4 combine with Theorem 1.1 to give information on cases (other than the six known cases) in which (1.1) has more than one solution. We hope this new information might eventually lead to a proof of the above conjecture.

This conjecture is a special case of the following conjecture given in [27]:

Conjecture 1.5. *Let $N(a, b, c)$ be the number of solutions in positive integers (x, y, z) to the equation*

$$a^x + b^y = c^z, a, b, c \in \mathbb{Z}^+, b > a > 1, \gcd(a, b) = 1, \tag{1.4}$$

with a, b, c not perfect powers.

If $N(a, b, c) > 1$, then (a, b, c) must be one of the following:

(i) $N(2, 2^r - 1, 2^r + 1) = 2$, $(x, y, z) = (1, 1, 1)$ and $(r + 2, 2, 2)$, where r is a positive integer with $r \geq 2$.

(ii) $N(2, 3, 11) = 2$, $(x, y, z) = (1, 2, 1)$ and $(3, 1, 1)$.

(iii) $N(2, 3, 35) = 2$, $(x, y, z) = (3, 3, 1)$ and $(5, 1, 1)$.

(iv) $N(2, 3, 259) = 2$, $(x, y, z) = (4, 5, 1)$ and $(8, 1, 1)$.

(v) $N(2, 5, 3) = 2$, $(x, y, z) = (1, 2, 3)$ and $(2, 1, 2)$.

(vi) $N(2, 5, 133) = 2$, $(x, y, z) = (3, 3, 1)$ and $(7, 1, 1)$.

(vii) $N(2, 7, 3) = 2$, $(x, y, z) = (1, 1, 2)$ and $(5, 2, 4)$.

(viii) $N(2, 89, 91) = 2$, $(x, y, z) = (1, 1, 1)$ and $(13, 1, 2)$.

(ix) $N(2, 91, 8283) = 2$, $(x, y, z) = (1, 2, 1)$ and $(13, 1, 1)$.

(x) $N(3, 5, 2) = 3$, $(x, y, z) = (1, 1, 3)$, $(1, 3, 7)$, and $(3, 1, 5)$.

(xi) $N(3, 10, 13) = 2$, $(x, y, z) = (1, 1, 1)$ and $(7, 1, 3)$.

(xii) $N(3, 13, 2) = 2$, $(x, y, z) = (1, 1, 4)$ and $(5, 1, 8)$.

(xiii) $N(3, 13, 2200) = 2$, $(x, y, z) = (1, 3, 1)$ and $(7, 1, 1)$.

An effective upper bound for $N(a, b, c)$ was first given by A. O. Gel'fond [8] (Mahler [17] had earlier shown that the number of solutions was finite, using his p-adic analogue of the Diophantine approximation method of Thue-Siegel, but his method is ineffective). A straightforward application of an upper bound on the number of solutions of binary S-unit equations due to F. Beukers and

H. P. Schlickewei [5] gives $N(a, b, c) \leq 2^{36}$. The following more accurate upper bounds for $N(a, b, c)$ have been obtained in recent years:

(i) (R. Scott and R. Styer [27]) If $2 \nmid c$ then $N(a, b, c) \leq 2$.

(ii) (Y. Z. Hu and M. H. Le [12]) If $\max\{a, b, c\} > 5 \cdot 10^{27}$, then $N(a, b, c) \leq 3$.

(iii) (Y. Z. Hu and M. H. Le [13]) If $2 \mid c$ and $\max\{a, b, c\} > 10^{62}$, then $N(a, b, c) \leq 2$.

(iv) (T. Miyazaki and I. Pink [18]) If $2 \mid c$ and $\max\{a, b, c\} \leq 10^{62}$, then $N(a, b, c) \leq 2$ except for $N(3, 5, 2) = 3$.

More recently, Miyazaki and Pink [19] have begun work on improving $N(a, b, c) \leq 2$ to $N(a, b, c) \leq 1$ under certain conditions, including some very specific results such as $c \notin \{2, 3, 6\}$ when $N(a, b, c) > 1$ ($c \neq 6$ has not previously been shown even for the more specific Pillai equation mentioned above).

Nevertheless, the problem of establishing $N(a, b, c) \leq 1$ with a finite number of specified exceptions remains open. This open question is addressed by the above Conjecture 1.5.

In Sections 3 and 4 we show that the first two cases given in Theorem 1.2 are impossible, and then, in Section 5, we prove Theorems 1.3 and 1.4. In Section 6, we consider (1.3) in the context of the *abc* conjecture.

2 Preliminary Lemmas

Lemma 2.1 (Theorem 6 of [25]). *Let p, q be distinct odd positive primes. For a given positive integer k, the equation*

$$q^n - p^m = 2^k, m, n \in \mathbb{N},$$

has at most one solution in positive integers (m, n).

Lemma 2.2 (Theorem 1.1 of [4]). *Let c and b be positive integers. Then there exists at most one pair of positive integers (z, y) for which*

$$0 < |c^z - b^y| < \frac{1}{4} \max\{c^{z/2}, b^{y/2}\}.$$

Lemma 2.3. *If a prime p is of the form $a^2 + 64b^2$ for some integers a and b, then 2 is a quartic residue modulo p.*

Proof. A proof is found in [7] which is simpler than Gauss's earlier proof of a conjecture of Euler. □

Lemma 2.4. *For any prime $p \equiv 1 \mod 16$, 2 is an octic residue modulo p if and only if $p = a^2 + 256b^2$ for some integers a and b.*

Proof. This lemma is found in Whiteman [32], who cites Reuschle [28] for the original statement of the lemma and Western [31] for the first proof. □

Lemma 2.5 (Theorem 1.8 of [16]). *If (1.1) has more than one solution and is not one of the six exceptional cases given in Theorem 1.1, then $p > 10^9$ and $q > 10^{18}$.*

3 $(p,q) \not\equiv (13,5) \bmod 24$

The purpose of this section is to prove the following

Proposition 3.1. *If $p \equiv q \equiv 5 \bmod 8$, then the equation*

$$2^x + p^y = q^z, p, q \in \mathbb{P}$$

has at most one solution in positive integers (x, y, z).

We will use four lemmas.

Lemma 3.2. *Let D be a natural number which is not a perfect square. Suppose the equation $h^2 - Dk^2 = -1$ is solvable, and that $h_1 + k_1\sqrt{D}$ is its fundamental solution. Let p be any prime dividing h_1. Then if $U^2 - V^2 D = 1$, we must have $p \mid V$.*

Proof. The lemma follows from Theorem 106 of [21] and Lemma 1 of [24]. □

For the next two lemmas we establish notation for the continued fraction for $\sqrt{D}$ and its convergents (the basic results on which this notation is based can be found in [23]). For any non-square positive integer D let

$$\sqrt{D} = [a_0, \overline{a_1, \ldots, a_s}]$$

represent the continued fraction expansion of $\sqrt{D}$. Let $\frac{P_m}{Q_m}$ be the m-th convergent of $\sqrt{D}$ and let

$$k_m = (-1)^{m+1}(P_m^2 - Q_m^2 D), \tag{3.1}$$

noting that (as shown in [23]) all the k_m are positive integers with

$$k_{ns+j} = k_j, j = 0, \ldots, s-1, n \in \mathbb{Z}^+. \tag{3.2}$$

We are now ready to state

Lemma 3.3 (Theorem 10.8.2 of [23]). *Let k be an integer. If $|k| < \sqrt{D}$ and (x, y) is a solution of $x^2 - y^2 D = k$ with $\gcd(x, yD) = 1$, then $\frac{|x|}{|y|}$ is a convergent of the continued fraction for $\sqrt{D}$.*

Lemma 3.4. *If $|x^2 - y^2 D| < \sqrt{D}$ and $\gcd(x, yD) = 1$, then $|x^2 - y^2 D| = k_m$ for some $m \leq s - 1$.*

Proof. By Lemma 3.3, x/y is a convergent of the continued fraction for $[a_0, \overline{a_1, \ldots, a_s}]$, so that (3.2) gives the lemma. □

The following lemma is obtained by direct calculation.

Lemma 3.5. *If $D = p^{2n} + 4$, where n is a positive integer, then we have*

$$\sqrt{D} = [p^n, \overline{(p^n - 1)/2, 1, 1, (p^n - 1)/2, 2p^n}]. \tag{3.3}$$

$$\frac{P_0}{Q_0} = \frac{p^n}{1}, \frac{P_1}{Q_1} = \frac{(p^{2n} - p^n + 2)/2}{(p^n - 1)/2}, \frac{P_2}{Q_2} = \frac{(p^{2n} + p^n + 2)/2}{(p^n + 1)/2},$$
$$\frac{P_3}{Q_3} = \frac{p^{2n} + 2}{p^n}, \frac{P_4}{Q_4} = \frac{(p^{2n} + 3)p^n/2}{(p^{2n} + 1)/2}. \tag{3.4}$$

$$k_0 = 4, k_1 = p^n, k_2 = p^n, k_3 = 4, k_4 = 1. \tag{3.5}$$

We are now ready to give

Proof of Proposition 3.1. Noting that (1.2) gives $\left(\frac{q}{p}\right) = 1$, let $\mathfrak{p}\bar{\mathfrak{p}}$ be the unique ideal factorization of p in $\mathbb{Q}(\sqrt{q})$, and let Z_1 be the least positive integer such that $\mathfrak{p}^{Z_1}$ is principal. $Z_1 \mid Z$ in any solution of $X^2 - Y^2 q = \pm p^Z$ with $\gcd(X, Yq) = 1$. Let θ be any integer of the field with norm $-p^{Z_1}$ such that $p \nmid \theta$. Let $\alpha = \theta^t$ where $t = \frac{y_2}{Z_1}$, where y_2 is as in (1.3). Since y_2 is odd, t is odd, so that α has norm $-p^{y_2}$. Let $\beta = 2^{x_2/2} + q^{(z_2-1)/2}\sqrt{q}$. By (1.3), β has norm $-p^{y_2}$. Now we have $\beta\bar{\beta} = \alpha\bar{\alpha}$, giving the equation in ideals

$$[\beta][\bar{\beta}] = [\alpha][\bar{\alpha}]$$

where $p \nmid \alpha$ and $p \nmid \beta$, so that $[\beta] = \mathfrak{p}^{y_2}$ or $[\beta] = \bar{\mathfrak{p}}^{y_2}$, and $[\alpha] = \mathfrak{p}^{y_2}$ or $[\alpha] = \bar{\mathfrak{p}}^{y_2}$. Thus, $[\beta] = [\alpha]$ or $[\beta] = [\bar{\alpha}]$ so there exists a unit δ such that

$$\beta = \delta\alpha, \text{ or } \beta = \delta\bar{\alpha}.$$

Let $\xi = \theta$ or $\bar{\theta}$ according as $\beta = \delta\alpha$ or $\delta\bar{\alpha}$. Thus we have

$$2^{x_2/2} + q^{(z_2-1)/2}\sqrt{q} = \xi^t \delta, \tag{3.6}$$

where δ has norm 1 since ξ^t has norm $-p^{y_2}$. Let

$$\theta = X_1 + Y_1\sqrt{q}, \delta = U + V\sqrt{q}, \tag{3.7}$$

where X_1, Y_1, U, and V are nonzero integers.

Now we apply Lemma 3.5 with $D = q$ and $n = y_1/2$, noting that, since $q \equiv 5 \mod 8$ and $2 \mid y_1$, we must have $x_1 = 2$ in (1.2). By (3.1), (3.4), and (3.5) we see that

$$P_2^2 - qQ_2^2 = -p^{y_1/2}. \tag{3.8}$$

Suppose $X^2 - Y^2 q = \pm p^{n_0}$ where $\gcd(X, Y) = 1$ and $n_0 < y_1/2$. Then by Lemma 3.4 we must have $p^{n_0} = k_m$ for some $m \le 4$, contradicting (3.5). So we have

$$Z_1 = y_1/2. \tag{3.9}$$

Since θ is any integer of the field with norm $-p^{Z_1}$ and with $p \nmid \theta$, by (3.8) and (3.4) we can take $\theta = X_1 + Y_1\sqrt{q}$ where

$$(X_1, Y_1, Z_1) = (\frac{1}{2}(p^{y_1} + p^{y_1/2} + 2), \frac{1}{2}(p^{y_1/2} + 1), y_1/2) \tag{3.10}$$

From (3.10) we see that
$$X_1^2 \equiv Y_1^2 q \equiv 1 \bmod p, \qquad (3.11)$$
where the last congruence holds since $q \equiv 4 \bmod p$, noting $x_1 = 2$ in (1.2).

By (3.4) and (3.5), we see that $P_4 + Q_4\sqrt{q}$ is the fundamental solution of $x^2 - y^2 q = -1$ (since P_m and Q_m increase with m). Since $p \mid P_4$, by Lemma 3.2 we must have $p \mid V$ in (3.7). Also $U^2 \equiv V^2 q + 1 \equiv 1 \bmod p$. So, in (3.7), we have
$$V \equiv 0 \bmod p, U \equiv \pm 1 \bmod p. \qquad (3.12)$$

By (3.6) we have
$$2^{x_2/2} + q^{(z_2-1)/2}\sqrt{q} = (X_1 \pm Y_1\sqrt{q})^t(U + V\sqrt{(q)}). \qquad (3.13)$$

Using (3.11) and considering the binomial expansion of $(X_1 \pm Y_1\sqrt{q})^t = X_t + Y_t\sqrt{q}$ we find
$$\pm Y_t \equiv 2^{t-1}Y_1 = 2^{t-2}(2Y_1) \equiv 2^{t-2} \bmod p, \qquad (3.14)$$
where the last congruence follows from (3.10). Since $q \equiv 4 \bmod p$, (3.13) and (3.12) give
$$2^{z_2-1} \equiv q^{(z_2-1)/2} \equiv X_t V + Y_t U \equiv \pm Y_t \bmod p. \qquad (3.15)$$

(3.14) and (3.15) give
$$2^{z_2-1} \equiv \pm 2^{t-2} \bmod p. \qquad (3.16)$$

Since z_2 and t are both odd and $p \equiv 1 \bmod 4$, (3.16) requires $\left(\frac{2}{p}\right) = 1$, contradicting $p \equiv 5 \bmod 8$, completing the proof of Proposition 3.1. $\qquad \square$

4 $\quad (p, q) \not\equiv (13, 17) \bmod 24$

The purpose of this section is to prove the following:

Proposition 4.1. *If $p \equiv 5 \bmod 8$ and $q \equiv 1 \bmod 8$, then the equation*
$$2^x + p^y = q^z, p, q \in \mathbb{P}$$
has at most one solution in positive integers (x, y, z).

We first prove a general lemma. We use the following notation: let d be a primitive root of a prime p; if an integer $n \equiv d^i \bmod p$ with $0 < i \le p - 1$, we call i the index of n for that primitive root d and write
$$i = i_p(n).$$

We also use the notation $v_2(n)$ to indicate the greatest integer t such that $2^t \mid n$. Notice that $v_2(\gcd(i_p(n), p - 1))$ is independent of the choice of primitive root d. For brevity, we use the following notation:
$$w_p(n) = v_2(\gcd(i_p(n), p - 1)),$$
so that $0 \le w_p(n) \le v_2(p - 1)$.

We use three simple observations:

Observation 1. *If $a \equiv b \bmod p$, then $w_p(a) = w_p(b)$.*

Observation 2. *We have*
(i) $w_p(-a) = w_p(a)$ when $w_p(a) < v_2((p-1)/2)$,
(ii) $w_p(-a) = v_2((p-1))$ when $w_p(a) = v_2((p-1)/2)$,
(iii) $w_p(-a) = v_2((p-1)/2)$ when $w_p(a) = v_2(p-1)$.

Observation 3. *For a given prime p and a given integer a, if $w_p(a^t) < v_2(p-1)$, then $v_2(t)$ is determined by $w_p(a^t)$.*

Lemma 4.2. *If (1.2) and (1.3) hold with $v_2(x_1) = v_2(x_2)$, then*

$$w_q(2^{x_1}) = v_2((q-1)/2). \tag{4.1}$$

Proof. We consider the solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) modulo q.

If $w_q(2^{x_1}) < v_2((q-1)/2)$, then, by Observation 2, $w_q(p^{y_1}) = w_q(2^{x_1})$. But then also, since $v_2(x_2) = v_2(x_1)$, we have $w_q(2^{x_2}) = w_q(2^{x_1}) = w_q(p^{y_1})$ and, by Observation 2, $w_q(2^{x_2}) = w_q(p^{y_2})$ so that $w_q(p^{y_1}) = w_q(p^{y_2})$. But this is impossible by Observation 3 since $2 \nmid y_1 - y_2$ and $w_q(2^{x_1}) = w_q(p^{y_1}) = w_q(p^{y_2}) < v_2((q-1)/2)$.

Similarly, if $w_q(2^{x_1}) > v_2((q-1)/2)$, then, by Observation 2, $w_q(p^{y_1}) = v_2((q-1)/2)$, and, since $w_q(2^{x_1}) = w_q(2^{x_2})$, also $w_q(p^{y_2}) = v_2((q-1)/2)$, which is impossible by Observation 3 since $2 \nmid y_1 - y_2$.

So we must have (4.1). $\square$

Proof of Proposition 4.1. Assume (1.1) has more than one solution with $p \equiv 5 \bmod 8$, $q \equiv 1 \bmod 8$. By Theorem 1.2, we can assume that $5 \nmid pq$. Considering congruences modulo 8, we find $2^{x_2} = 4$. Since 2 is a quadratic nonresidue of p, we have $w_p(2) = 0$, so that, by Observation 1, we have $1 = w_p(4) = w_p(2^{x_2}) = w_p(q^{z_2}) = w_p(q) = w_p(2^{x_1})$, so that, by Observation 3,

$$v_2(x_1) = v_2(x_2) = 1, \tag{4.2}$$

noting that $w_p(q^{z_2}) = w_p(q)$ since z_2 is odd. So we can apply Lemma 4.2 to obtain (4.1).

From (4.2) we have $v_2(x_1) = 1$, so that $2^{x_1} \equiv 4 \bmod 5$. Since $5 \nmid pq$ and $2 \mid y_1$, we must have $p^{y_1} \equiv 4 \bmod 5$. So $2 \| y_1$, giving

$$p^{y_1} \equiv 9 \bmod 16. \tag{4.3}$$

Since $q \equiv 1 \bmod 8$, $2^{x_1} > 4$, which, along with (4.2), gives

$$2 \| x_1 > 2. \tag{4.4}$$

Considering congruences modulo 16 and using (1.2), (4.3), and (4.4), we see that $2^3 \| q - 1$, so that $v_2((q-1)/2) = 2$, and (4.1) becomes

$$w_q(2^{x_1}) = v_2((q-1)/2) = 2. \tag{4.5}$$

From (4.4) we see that $q = p^{y_1} + 2^{x_1}$ is of the form $a^2 + 64b^2$, so, by Lemma 2.3, 2 is a quartic residue modulo q, so that $4 \mid i_q(2)$ for any choice of primitive root d. Thus $8 \mid i_q(2^{x_1})$ so that

$$w_q(2^{x_1}) > 2,$$

contradicting (4.5), proving Proposition 4.1. □

5 Proofs of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. From Theorem 1.2, Proposition 3.1, and Proposition 4.1, we have

$$(p, q) \equiv (1, 17) \bmod 24. \tag{5.1}$$

We prove (i), (ii), and (iii) separately.

(i): From (1.2) and (1.3) we have

$$2^{x_1} + (p^{y_1} - 1) = (q - 1), 2 \mid x_1, 2 \mid y_1,$$

and

$$2^{x_2} + (p^{y_2} - 1) = (q^{z_2} - 1), 2 \mid x_2, 2 \nmid y_2, 2 \nmid z_2.$$

If $v_2(p - 1) > v_2(q - 1)$, then we see that $x_1 = v_2(q - 1) = v_2(q^{z_2} - 1) = x_2$, contradicting Lemma 2.1, so that

$$v_2(p - 1) \le v_2(q - 1). \tag{5.2}$$

So now we have either

$$v_2(p - 1) = v_2(q - 1) = x_1 \tag{5.3}$$

or

$$v_2(q - 1) > v_2(p - 1) = x_2. \tag{5.4}$$

Since $2 \mid x_1$ and $2 \mid x_2$, from (5.3) and (5.4) we have

$$2 \mid v_2(p - 1). \tag{5.5}$$

From (5.1) and (5.5) we have

$$v_2(p - 1) \ge 4, \tag{5.6}$$

which, in combination with (5.1) and (5.2), gives

$$p \equiv 1 \bmod 48, q \equiv 17 \bmod 48. \tag{5.7}$$

(5.7), (5.5), and (5.2) give (i) of Theorem 1.3.

(ii): We use the notation of Sections 1 and 4. Using Observation 1 of Section 4 and noting that z_2 is odd, we find

$$w_p(2^{x_1}) = w_p(q) = w_p(q^{z_2}) = w_p(2^{x_2}).$$

If $u_p(q)$ is even, then $w_p(q) \le v_2(\frac{p-1}{2})$, so that $w_p(2^{x_1}) = w_p(2^{x_2}) \le v_2(\frac{p-1}{2})$, so that, by Observation 3 of Section 4,

$$v_2(x_1) = v_2(x_2). \tag{5.8}$$

So we can apply Lemma 4.2 to find

$$w_q(2^{x_1}) = v_2(\frac{q-1}{2}).$$

So also, by (5.8), $w_q(2^{x_2}) = v_2(\frac{q-1}{2})$, so that by (ii) of Observation 2 of Section 4, $w_q(p^{y_2}) = w_q(p) = v_2(p-1)$, so that $u_q(p)$ is odd, giving (ii) of Theorem 1.3.

(iii): Assume that we do not have $v_2(p-1) = v_2(q-1) = 4$. We can also assume we do not have $v_2(p-1) = v_2(q-1) = 6$ since then (1.2) becomes $2^6 + p^{y_1} = q$ so that, by Lemmas 2.3 and 2.4, $w_q(2) = 2$ and $w_q(2^{x_1}) = w_q(2) + v_2(6) = 3$; applying Observation 2(i) to equations (1.2) and (1.3) and noting that $v_2(y_1) > v_2(y_2)$, we find $3 = w_q(p^{y_1}) > w_q(p^{y_2}) = w_q(2^{x_2})$, requiring $2 \nmid x_2$, contradicting (1.3).

So now, if $v_2(p-1) = v_2(q-1)$, $x_1 \ge 8$ (by (5.5)). And if $v_2(p-1) < v_2(q-1)$, then $x_2 = v_2(p-1)$ and $x_1 > v_2(p-1)$, so that, by (5.5) and (5.6), $x_1 \ge 8$, unless $x_2 = 4$ and $x_1 = 6$ which is an impossible case by Lemmas 2.2 and 2.5. So $x_1 \ge 8$, so that q is of the form $a^2 + 256b^2$. By (5.7), $q \equiv 1 \bmod 16$. So (iii) of Theorem 1.3 follows from Lemma 2.4. $\qquad\square$

Proof of Theorem 1.4. By Corollary 1.6 of [16], $z_2 > 1$ in (1.3). So, using Lemma 2.5, we have

$$2^{27} < \frac{10^9}{4} < \frac{q^{1/2}}{4}, 2^{87} < \frac{10^{27}}{4} < \frac{q^{3/2}}{4} \le \frac{q^{z_2/2}}{4},$$

so that Lemma 2.2 applies to give (i) of Theorem 1.4. So (5.3) and (5.4) give (ii) of Theorem 1.4. $\qquad\square$

6 Unlikelihood of Equation (1.3)

In this section we consider the equation (1.3) in the context of the *abc* conjecture. Let a, b, and c be positive integers such that $a + b = c$; define $Q = Q(a, b, c)$ as

$$Q = \frac{\log(c)}{\log(\mathrm{rad}(abc))}$$

where $\mathrm{rad}(m)$ is the product of all distinct primes dividing m. Then for (1.3) we have

$$Q = \frac{z_2 \log(q)}{\log(2) + \log(p) + \log(q)} \ge \frac{3 \log(q)}{(3/2) \log(q) + \log(2)}$$

$$= 2 - \frac{2 \log(2)}{(3/2) \log(q) + \log(2)} > 1.97.$$

The highest value for Q found in recent researches on the *abc* conjecture is $Q = 1.62991$ for $(a, b, c) = (2, 3^{10} \cdot 109, 23^5)$. If $z_2 > 3$, then we have $Q > 3.29$: if a conjecture of Tenenbaum (quoted in Section B19 of [9]) is true, then $Q = 3.29$ is impossible, so that $z_2 = 3$.

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