The equation $|p^x \pm q^y| = c$ in nonnegative $x$, $y$.

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Abstract

We improve earlier work on the title equation (where $p$ and $q$ are primes and $c$ is a positive integer) by allowing $x$ and $y$ to be zero as well as positive. Earlier work on the title equation showed that, with listed exceptions, there are at most two solutions in positive integers $x$ and $y$, using elementary methods. Here we show that, with listed exceptions, there are at most two solutions in nonnegative integers $x$ and $y$, but the proofs are dependent on nonelementary work of Mignotte, Bennett, Luca, and Szalay.

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1 Introduction

Earlier work ([2], [8], [13], [14], [15]) has treated the equation

$$(−1)^u a^x + (−1)^v b^y = c$$

for integers $a > 1$, $b > 1$, $c > 0$, with solutions $(x, y, u, v)$ where $u, v \in \{0, 1\}$ and $x$ and $y$ are positive integers. Recently, in treating the more general Pillai equation

$$(−1)^u r a^x + (−1)^v s b^y = c$$ (P)

(where $r$ and $s$ are positive integers; see [7], [8], [16], [17], [20], [22]), the authors noticed that it is in a sense a more natural approach to allow $x$ and $y$ to be zero as well as positive; this is because analyzing (P) is greatly clarified by the use of what the authors in [17] call basic forms, which require exponents equal to zero (see Lemma 1.1 of [17] for a definition of basic form).

So in this paper we improve earlier results (Theorems 3 and 5 of [13] and Theorem 7 of [14]) by allowing the variables $x$ and $y$ in $$(−1)^u a^x + (−1)^v b^y = c$$ to be zero as well as positive, which significantly alters the nature of the proofs: while the proofs in [13] and [14] are elementary, the proofs of Theorems 1, 2, and 3 below depend on non-elementary work of Luca [9] and Szalay [21], and the proof of Theorem 3 depends also on non-elementary results of Mignotte [10] and Bennett [2]. We have not been able to remove dependence on these non-elementary results, but we have been able to replace the proofs in [9] and some of the proofs in [21] by short elementary proofs, thus making Theorems 1 and 2 elementary. (For this reason we state Theorem 2 and the nonelementary Theorem 3 separately even though Theorem 3 includes Theorem 2 except for the trivial case $p = q$.) For the most part, we restrict the bases $a$ and $b$ to prime values. Computer searches in [2] and [15] (supplemented with calculations on the second author’s website) suggest that the list of exceptions in Theorem 3 below would remain unchanged even if $p$ and $q$ were allowed to be any relatively prime integers (here of course we would be redefining $F$ and $M$ to allow composite Fermat and Mersenne numbers).

We give the following results:
Theorem 1. For integers \( b > 1 \) and \( c > 0 \) and positive prime \( a \), the equation
\[
a^x - b^y = c
\]
has at most two solutions in nonnegative integers \((x, y)\), except for \((a, b, c) = (2, 5, 3)\), which has solutions \((x, y) = (2, 0), (3, 1), (7, 3)\).

There are an infinite number of \((a, b, c)\) for which (1) has two solutions.

Theorem 2. For positive primes \( p \) and \( q \) and positive integer \( c \), the equation
\[
|p^x - q^y| = c
\]
has at most two solutions in nonnegative integers \( x \) and \( y \), except when \((p, q, c)\) or \((q, p, c)\) is equal to \((3, 5, 2), (2, 3, 5), (2, 3, 7), (2, 11, 7), (2, F, F - 2)\) where \( F \) is a Fermat prime.

Theorem 3. For distinct positive primes \( p \) and \( q \) and positive integer \( c \) there are at most two solutions to the equation
\[
(-1)^a p^x + (-1)^b q^y = c
\]
in nonnegative integers \( x \) and \( y \) and integers \( u, v \in \{0, 1\} \), except when \((p, q, c)\) or \((q, p, c)\) is equal to one of the following: \((2, 3, 1), (2, 3, 5), (2, 3, 7), (2, 3, 11), (2, 3, 13), (2, 3, 17), (2, 5, 3), (2, 5, 7), (2, 5, 9), (2, 11, 7), (3, 5, 2), (3, 5, 4), (3, 13, 10), (2, F, F - 2), (2, F, 2F - 1), (2, M, M + 2), (2, M, 2M + 1), (3, 3^n + (-1)^\delta 2, 2), (2, 2^\delta + (-1)^\delta 3, 3)\) where \( F > 5 \) is a Fermat prime, \( M > 3 \) is a Mersenne prime, \( \delta \in \{0, 1\}, n > 1 \) is a positive integer such that \((n, \delta) \neq (3, 1)\), and \( t > 1 \) is a positive integer such that \((t, \delta) \neq (2, 1), (3, 1), (7, 1)\).

The solutions in these cases are as follows:
\[
\begin{align*}
2 - 1 &= -2 + 3 = 2^2 - 3 = -2^3 + 3^2 = 1 \\
2^2 + 1 &= 2 + 3 = 2^3 - 3 = -2^2 + 3^2 = 2^5 - 3^3 = 5 \\
2^3 - 1 &= 2^2 + 3 = -2 + 3^2 = 2^4 - 3^2 = 7 \\
2^3 + 3 &= 2 + 3^2 = -2^4 + 3^3 = 11 \\
2^2 + 3^2 &= 2^4 - 3 = 2^8 - 3^4 = 13 \\
2^4 + 1 &= 2^3 + 3^2 = 2^6 + 3^4 = 17 \\
2 + 1 &= 2^2 - 1 = 6 + 5 = 2^3 - 5 = 2^7 - 5^3 = 3 \\
2^3 - 1 &= 2^2 + 5 = 2^3 - 5^2 = 7 \\
2^3 + 1 &= 2^2 + 5 = 7^2 + 5^3 = 9 \\
2^3 - 1 &= 2^2 + 11 = 2^7 - 11^2 = 7 \\
1 + 1 &= 3 - 1 = -3 + 5 = 3^3 - 5^2 = 2 \\
3 + 1 &= -1 + 5 = 3^2 - 5 = 4 \\
3^2 + 1 &= 3 + 13 = -3^7 + 13^3 = 10 \\
(F - 1) - 1 &= -2 + F = (2F - 2) - F = F - 2 \\
(2F - 2) + 1 &= (F - 1) + F = -(F - 1)^2 + F^2 = 2F - 1 \\
(M + 1) + 1 &= 2 + M = (2M + 2) - M = M + 2 \\
(2M + 2) - 1 &= (M + 1) + M = (M + 1)^2 - M^2 = 2M + 1 \\
1 + 1 &= 3 - 1 = -(-1)^\delta 3^n + (-1)^\delta (3^n + (-1)^\delta 2) = 2 \\
2 + 1 &= 2^2 - 1 = -(-1)^\delta 2^t + (-1)^\delta (2^t + (-1)^\delta 3) = 3
\end{align*}
\]

Theorem 1 is proven in [19]. Theorems 2 and 3 are proven in Sections 3 and 4, respectively, of this paper.

We are grateful to Michael Bennett for proving \( y_3 = 1 \) in equations (75) and (76) below by pointing out references [3] and [4].
Preliminary Lemmas

Lemma 1. ([9]) The only solutions to the equation

$$p^r \pm p^s + 1 = z^2$$

in positive integers $(z, p, r, s)$ with $r > s$ and $p$ an odd prime are $(z, p, r, s) = (5, 3, 3, 1), (11, 5, 3, 1)$ with the above sign taken as minus.

Proof. See [9] or, for a short elementary proof, see [19].

The following lemma proven in [21] can be made elementary simply by replacing the result from [6] used in Szalay’s proof by the elementary result [5, Theorem 4].

Lemma 2. ([21]) The equation

$$2^r - 2^s + 1 = z^2$$

has no solutions in positive integers $(r, s, z)$ with $r > s$ except for the following cases:

$$(r, s, z) = (2t, t + 1, 2^t - 1)$$
for positive integer $t > 1$

$$(r, s, z) = (5, 3, 5)$$

$$(r, s, z) = (7, 3, 11)$$

$$(r, s, z) = (15, 3, 181)$$

Lemma 2 is the only result from [21] which we will need for Theorem 2. For Theorem 3 we will use a further result from [21] for which we have not found a purely elementary proof. However, we do give a shorter simpler proof:

Lemma 3. ([21]) The equation

$$2^r + 2^s + 1 = z^2$$

has no solutions in positive integers $(r, s, z)$ with $r \geq s$ except for the following cases:

$$(r, s, z) = (2t, t + 1, 2^t + 1)$$
for positive integer $t$  

$$(r, s, z) = (5, 4, 7)$$

$$(r, s, z) = (9, 4, 23)$$

Proof. Assume (4) has a solution that is not one of (5), (6), or (7). It is an easy elementary result that the only solution to (4) with $r = s$ is given by Case (5) with $t = 1$, so we can assume hereafter $r > s$.

Considering (4) modulo 8, we get $s > 2$. If $s = 3$, then $2^r = z^2 - 2^3 - 1 = (z + 3)(z - 3)$, giving $z = 5$, which is Case (5) with $t = 2$, so we can assume hereafter $s > 3$.

Write $z = 2^t k \pm 1$ for $k$ odd and the sign chosen to maximize $t > 1$. In what follows, we will always take the upper sign when $z \equiv 1 \pmod{4}$ and the lower sign when $z \equiv 3 \pmod{4}$.

We have

$$2^r + 2^s + 1 = 2^{2t}k^2 \pm (k \mp 1)2^{t+1} + 2^{t+1} + 1.$$  

(8)
From this we see \( s = t + 1 \) so that \( t \geq 3 \). Now (8) yields \( r \geq 2t - 1 \) with equality only when \( t = 3, k = 1 \), and \( z \equiv 3 \mod 4 \), which is Case (6), already excluded. So \( r \geq 2t \), hence \( r > 2t \) since Case (5) has been excluded. So now
\[
k \equiv 1 \mod 2t \quad \text{for some odd } g > 0.
\]
We have
\[
2^r - 2t = k^2 \pm g = 2^{2t} - 2g^2 \pm 2^t g + 1 \pm g.
\] (9)
(9) yields \( r - 2t \geq 2t - 3 \) with equality only when \( t = 3, g = 1 \), and \( z \equiv 3 \mod 4 \), which is Case (7), already excluded. So now \( g \equiv 1 \mod 2t \) for some odd \( h > 0 \). We have
\[
2^r - 2t = k^2 \pm g = 2^{2t} - 2g^2 \pm 2^t g + 1 \pm g.
\]
(9) yields \( r - 2t \geq 2t - 3 \) with equality only when \( t = 3, g = 1 \), and \( z \equiv 3 \mod 4 \), which is Case (7), already excluded. So now \( g \equiv 1 \mod 2t \) for some odd \( h > 0 \). We must have \( g \geq 2t + 1 \). Assume \( z \equiv 3 \mod 4 \). Then from (9) we derive
\[
2^r - 2t \geq 2^t g^2 = 2^{2t} - 2g^2 \pm 2^t g + 1 \pm g.
\]
Now assume \( z \equiv 1 \mod 4 \). Then
\[
2^r - 2t \geq 2^{2t} - 2g^2 = 2^{2t} - 2g^2 \pm 2^t g + 1 \pm g.
\]
In both cases we have
\[
r \geq 6t - 2 = 6s - 8.
\] (11)
Now we can use Corollary 1.7 in Bauer and Bennett [1]:
\[
r < \frac{2}{2 - 1.48} \frac{\log(2^s + 1)}{\log(2)}
\]
Thus,
\[
r < \frac{1}{0.26} \frac{\log(2^s + 1)}{\log(2)} s < \frac{1}{0.26} \frac{\log(17)}{\log(16)} s < 4s.
\]
Combining this with (11) we obtain \( s < 4 \) which is impossible since \( s > 3 \).

3 Proof of Theorem 2

Write \( v_a(b) \) to mean the highest power of \( a \) dividing \( b \) for positive prime \( a \) and nonzero integer \( b \); thus, \( a^{v_a(b)} || b \).

Proof of Theorem 2: Clearly three solutions are impossible if \( p = q \), so we can assume \( p \) and \( q \) are distinct primes. Excluding the exceptions listed in the theorem, assume we have more than two solutions to (2). Clearly there is at most one solution for which \( \min(x, y) = 0 \). Noting that the exceptional cases of Theorem 5 of [13] have been excluded, we can assume we have exactly one solution in which \( \min(x, y) = 0 \) and exactly two further solutions. After Theorem 1 above, we see that, without loss of generality, it suffices to consider just two cases.

Case 1: Assume (2) has exactly three solutions in the following form:
\[
q^{x_1} + c = p^{y_1},
\]
\[
p^{x_2} + c = q^{y_2},
\]
\[
1 + c = p^{x_3},
\]
where \( x_i > 0 \) and \( y_j > 0 \) for \( 1 \leq i \leq 3, 1 \leq j \leq 2 \).
Consideration modulo 2 gives \( q > 2 \). Assume first also \( p > 2 \). Substituting (14) into (12) and (13) we get \( q^{p_1} \equiv 1 \mod p \) and \( q^{p_2} \equiv -1 \mod p \), so \( 2 \mid y_1 = 2k \) for some positive integer \( k \). But then

\[
q^{2k} = p^{x_1} - c = p^{x_1} - p^{x_3} + 1,
\]

contradicting Lemma 1 unless \((p, q, c) = (3, 5, 2)\) or \((5, 11, 4)\). The case \((3, 5, 2)\) has been excluded and the case \((5, 11, 4)\) makes (13) impossible modulo 11.

So we can assume \( p = 2 \). If \( x_3 = 1 \), then \( c = 1 \), and it is a familiar elementary result that we must have \( q = 3 \), giving an excluded case. So we can assume \( x_3 \geq 2 \) and also \( x_1 \geq 3 \).

If \( x_2 \geq 2 \), then, substituting (14) into (12) and (13) we get \( q^{p_1} \equiv 1 \mod 4 \) and \( q^{p_2} \equiv 3 \mod 4 \), so that \( 2 \mid y_1 \), violating Lemma 2 unless \( q^{p_1/2} = 2^{x_3-1} - 1 \) for \( x_3 > 3 \), or \( c = 7 \) with \( q = 3, 5, 11, \) or 181. Since we have \( q \equiv 3 \mod 4 \) and have excluded the cases \((p, q, c) = (2, 3, 7)\) and \((2, 11, 7)\), we are left with \( x_3 > 3, y_1 = 2 \), and \( q = 2^{x_3-1} - 1 \) (noting the familiar elementary result that \( q^{p_1/2} \) cannot be a perfect power). In this case, \((\frac{q}{2}) = \left(\frac{2^{x_3-1} - 1}{q}\right) = 1\), making (13) impossible since also \((\frac{q}{2}) = 1\) and \( q \equiv 3 \mod 4 \).

It remains to consider \( x_2 = 1 \), in which case \( q^{p_2} = 2^{x_3+1} + 1 \). \( y_2 > 1 \) requires \( q^{p_2} = 9 \) giving \((p, q, c) = (2, 3, 7)\) which has already been excluded. So \( q^{p_2} = q = F \), a Fermat prime, giving the final exceptional case in the formulation of Theorem 2 (note the case \( x_3 = 1 \) has already been dealt with). This completes the proof of Case 1.

**Case 2:** Assume (2) has exactly three solutions in the following form:

\[
p^{x_1} + c = q^{y_1}, \tag{15}
\]

\[
p^{x_2} + c = q^{y_2}, \tag{16}
\]

\[
1 + c = p^{x_3}, \tag{17}
\]

where \( x_i > 0 \) and \( y_j > 0 \) for \( 1 \leq i \leq 3, 1 \leq j \leq 2 \).

By Theorem 3 of [13] we have \( 2 \not| x_1 - x_2 \), noting that the exceptional cases of Theorem 3 of [13] for which \( c \leq 5 \) have been excluded, while the exceptional cases of Theorem 3 of [13] for which \( c > 5 \) do not allow a solution with \( \min(x, y) = 0 \). Consideration modulo 2 gives \( q > 2 \).

Assume first \( p > 2 \). Substituting (17) into (15) and (16) we find \( q^{y_1} \equiv q^{y_2} \equiv -1 \mod p \), so that \( v_2(y_1) = v_2(y_2) \). So \( v_2(q^{y_1} - 1) = v_2(q^{y_2} - 1) \). Now rewrite (15) and (16) as

\[
(p^{x_1} - 1) + c = (q^{y_1} - 1), \tag{18}
\]

\[
(p^{x_2} - 1) + c = (q^{y_2} - 1). \tag{19}
\]

If \( v_2(c) = v_2(q^{y_1} - 1) = v_2(q^{y_2} - 1) \), then \( v_2(p^{x_1} - 1) > v_2(c) \) and \( v_2(p^{x_2} - 1) > v_2(c) \). But then, since at least one of \( x_1 \) and \( x_2 \) is odd, we get \( v_2(p - 1) > v_2(c) \), contradicting (17). On the other hand, if \( v_2(c) \neq v_2(q^{y_1} - 1) \), then we must have \( v_2(p^{x_1} - 1) = v_2(p^{x_2} - 1) \), violating \( 2 \not| x_1 - x_2 \).

So we must have \( p = 2 \). Recalling \( 2 \not| x_1 - x_2 \), take \( 2 \not| x_1, 2 \mid x_2 \). Consideration modulo 3 gives \( q \equiv 2 \mod 3, 2 \not| y_1, 2 \mid y_2 \). Now (16) gives \( c \equiv 1 \mod 4 \), so that (17) gives \( c = 1 \), and it is a familiar elementary result that we must have \( q = 3 \), giving an excluded case.

## 4 Proof of Theorem 3

We will use the following lemma based on a result of Mignotte [10] as used by Bennett [2].
Lemma 4. Let \( a > 1, b > 1, c > 1, x > 0, \) and \( y > 0 \) be integers such that \( (a, b) = 1 \) and
\[ a^x - b^y = c. \]

Let \( G = y/\log(a) \). Then either
\[ G < 2409.08 \] (20)
or
\[ G < \frac{2\log(c)}{\log(a) \log(b)} + 22.997(\log(G) + 2.405)^2. \] (21)

Also when \( G = x/\log(b) \) we have (20) or (21).

Proof. When \( G = x/\log(b) \) the lemma can be derived in essentially the same way as Equation (11) of [15]. Now assume both (20) and (21) fail to hold for \( G = y/\log(a) \), so that (20) fails to hold for \( G = x/\log(b) \).

But if (21) fails to hold for \( G = y/\log(a) \geq 2409.08 \), it must also fail to hold for any \( G > y/\log(a) \), so that (21) fails to hold for \( G = x/\log(b) \), a contradiction since we have shown at least one of (20) or (21) must hold for \( G = x/\log(b) \). \(\square\)

Proof of Theorem 3: We will first show that the exceptional \((p, q, c)\) listed in the formulation of Theorem 3 are the only \((p, q, c)\) which could have three or more solutions to (3); then, at the end of the proof, we will find all solutions \((x, y)\) for these \((p, q, c)\).

The exceptional cases of Theorems 3 and 4 of [13], Theorem 7 of [14], and Theorem 2 of the present paper are all included in the list of exceptions in the formulation of Theorem 3 above. So in what follows we will use all these results without explicitly dealing with the exceptional \((p, q, c)\).

Note that (3) can have at most two solutions with \( \min(x, y) = 0 \).

We first handle the cases \((p, q) = (2, 3), (3, 2), (2, 5), \) and \((5, 2)\). If one of these cases gives three solutions to (3), then \( c \) is odd and there is at most one solution with \( \min(x, y) = 0 \), unless \( c = 3 \) which gives the excluded case \((p, q, c) = (2, 5, 3)\) listed in the formulation of the theorem. So when (3) has more than two solutions with \( \min(p, q) = 2 \) and \( \max(p, q) \in \{3, 5\} \), we can assume we have at least two solutions for which \( \min(x, y) > 0 \). Now Theorem 4 of [13] and Pillai’s results in [12] suffice to give all \((p, q, c)\) such that \((p, q) = (2, 3)\) or \((3, 2)\) and (3) has at least two solutions for which \( \min(x, y) > 0 \), and it is easily determined which of these \((p, q, c)\) give more than two solutions to (3) in nonnegative integers \( x \) and \( y \); we list such \((p, q, c)\) in the formulation of Theorem 3. The methods of Pillai [12] can be used in just the same way to handle the case \((p, q) = (2, 5)\) or \((5, 2)\), so that, again using also Theorem 4 of [13], we can list all \((p, q, c)\) such that \((p, q) = (2, 5)\) or \((5, 2)\) and (3) has more than two solutions. So from here on we will assume
\[ p = 2 \implies q > 5, q = 2 \implies p > 5. \] (22)

Also, in the following search for \((p, q, c)\) allowing three or more solutions to (3), we will exclude all the exceptional cases listed in Theorem 3 from consideration.

After Theorem 7 of [14] and Theorem 2 of the present paper it suffices to consider only cases in which (3) has three solutions at least one of which has \( \min(x, y) = 0 \) and at least one of which has \((u, v) = (0, 0)\).

We divide the proof into thirteen such cases which can be seen to include all possibilities. In each of these cases, \( p \geq 2 \) and \( q \geq 2 \) are distinct primes unless otherwise indicated (in the first three cases we specify \( \min(p, q) > 2 \)). In the first nine cases, we assume exactly one of the exponents \( \{x_1, x_2, x_3, y_1, y_2, y_3\} \) is zero and the rest are positive. In the final four cases, more than one of the exponents is zero.

Note: in all thirteen cases the explicitly written exponents \( x_i \) and \( y_j \) are assumed to be greater than zero \((1 \leq i \leq 3, 1 \leq j \leq 3)\). Terms with exponent zero are written simply as “1”.

6
Case 1

\[ 1 + c = q^{y_1} \quad (23) \]
\[ p^{x_2} + q^{y_2} = c \quad (24) \]
\[ q^{y_3} + c = p^{x_3} \quad (25) \]

where \( p \) and \( q \) are odd primes. Substituting (23) into (24) and (25), we find \( q \mid p^{x_2} + 1 \) and \( q \mid p^{x_3} + 1 \), so that \( v_2(x_2) = v_2(x_3) \), giving \( p^{x_2} \equiv p^{x_3} \mod 4 \). So

\[ q^{y_3} = p^{x_2} - c \equiv p^{x_2} - c = -q^{y_2} \mod 4, \]

so

\[ q \equiv 3 \mod 4, 2 \nmid y_3 - y_2. \quad (26) \]

From (23) we have \( \left( \frac{q}{p} \right) = -1 \), so that, from (24) and (25),

\[ \left( \frac{p}{q} \right) = -1. \quad (27) \]

If \( p \equiv 3 \mod 4 \), then (27) requires \( \left( \frac{q}{p} \right) = 1 \) so (25) requires \( \left( \frac{q}{p} \right) = -1 \) while (24) requires \( \left( \frac{q}{p} \right) = \left( \frac{q^{y_2}}{p} \right) = \left( \frac{q^{y_3}}{p} \right) \), so that \( 2 \mid y_3 - y_2 \), contradicting (26).

Case 2

\[ 1 + q^{y_1} = c \quad (28) \]
\[ p^{x_2} + q^{y_2} = c \quad (29) \]
\[ q^{y_3} + c = p^{x_3} \quad (30) \]

where \( p \) and \( q \) are odd primes. Substituting (28) into (29) and (30), we find that, by Lemma 1, \( x_2 \) is odd unless \( (p, q, c) = (5, 3, 28) \) or \( (11, 5, 126) \), and \( x_3 \) is odd, making \( (p, q, c) = (5, 3, 28) \) impossible modulo 3; also (30) is impossible modulo 11 if \( (p, q, c) = (11, 5, 126) \). So we can assume \( x_2 \) and \( x_3 \) are both odd. Rewrite (29) and (30) as

\[ (p^{x_2} - 1) + (q^{y_2} + 1) = c \quad (31) \]

and

\[ (q^{y_3} - 1) + c = p^{x_3} - 1. \quad (32) \]

Since \( x_2 \) and \( x_3 \) are both odd, \( v_2(p^{x_2} - 1) = v_2(p^{x_3} - 1) \). Suppose \( v_2(p^{x_2} - 1) < v_2(c) \). Then we must have, from (31) and (32), \( v_2(q^{y_2} + 1) = v_2(p^{x_2} - 1) = v_2(p^{x_3} - 1) = v_2(q^{y_3} - 1) \); this is possible only if \( q \equiv 3 \mod 4 \) and \( v_2(q^{y_2} + 1) = v_2(q^{y_3} - 1) = 1 \) so we must have \( v_2(p^{x_2} - 1) = v_2(p^{x_3} - 1) = 1 \). So now write equations (29) and (30) as

\[ (p^{x_2} + 1) + (q^{y_2} - 1) = c \quad (33) \]

and

\[ (q^{y_3} + 1) + c = p^{x_3} + 1. \quad (34) \]
Note that in both (33) and (34) all three terms have valuation base 2 greater than 1 when \( v_2(p^{x_2} - 1) < v_2(c) \). Therefore, \( y_1 \) and \( y_3 \) are both odd so that \( v_2(c) = v_2(q^{y_3} + 1) \). Therefore, from (34), we have \( v_2(p^{x_3} + 1) > v_2(c) \) and since \( v_2(p^{x_3} + 1) = v_2(p^{x_2} + 1) \), we have \( v_2(p^{x_2} + 1) > v_2(c) \). But we must also have \( y_2 \) even and \( y_1 \) odd so that \( v_2(q^{y_2} - 1) > v_2(c) \). Thus (33) becomes impossible, eliminating the possibility \( v_2(p^{x_2} - 1) < v_2(c) \).

Now suppose \( v_2(c) < v_2(p^{x_2} - 1) = v_2(p^{x_3} - 1) \). Now from (31) and (32) we see that \( v_2(c) = v_2(q^{y_2} + 1) = v_2(q^{y_3} - 1) = 1 \). Now write (29) and (30) as

\[
(p^{x_2} - 1) + (q^{y_2} - 1) = (c - 2)
\]

and

\[
(q^{y_3} + 1) + (c - 2) = p^{x_3} - 1.
\]

Note that in both (35) and (36) all three terms have a valuation base 2 greater than 1. We must have \( q \equiv 3 \mod 4 \) with \( v_2(q^{y_3} + 1) < v_2(q^{y_1} - 1) = v_2(c - 2) \), so that, from (36), \( v_2(c - 2) > v_2(p^{x_3} - 1) = v_2(p^{x_2} - 1) \), so that \( v_2(q^{y_3} + 1) = v_2(p^{x_3} - 1) = v_2(p^{x_2} - 1) = v_2(q^{y_2} - 1) \), which is impossible. This eliminates the possibility \( v_2(c) < v_2(p^{x_2} - 1) \).

So we are left with \( v_2(c) = v_2(p^{x_2} - 1) = v_2(p^{x_3} - 1) \). In this case from (31) we see that \( v_2(q^{y_2} + 1) > v_2(c) \) so that \( q \equiv 3 \mod 4 \) and \( v_2(c) = v_2(q^{y_1} + 1) = 1 \). From (32) we see that \( v_2(q^{y_3} - 1) > 1 \). So we have

\[
2 \mid y_1, 2 \mid y_2, 2 \mid y_3, 2 \not| x_2, 2 \not| x_3.
\]

Recalling (28) and using (30) we see that consideration modulo 8 gives \( p \equiv 3 \mod 8 \) so that (29) gives \( q \equiv 7 \mod 8 \), so that \( q \neq 3 \). Now consideration modulo 3 gives (recalling (28) and using (30)) \( p = 3 \); also (recalling (29)) \( q \equiv 2 \mod 3 \). To handle this case we make the following substitutions into Lemma 4 (noting \( c > 1 \)): \( a = 3, b = q, x = x_3, y = y_3 \). We get either

\[
\frac{y_3}{\log(3)} < 2409.08
\]

or

\[
\frac{y_3}{\log(3)} < \frac{2 \log(c)}{\log(3) \log(q)} + 22.997(\log(y_3) - \log(3) + 2.405)^2.
\]

From (28) and (29) we have \( y_1 > y_2 \). From (29) and (30) we have \( x_3 > x_2 \). By Lemma 2.11 of [17] we must have

\[
y_2 < y_1 < y_3,
\]

noting that none of the exceptional cases of Lemma 2.11 of [17] fits Case 2.

Combining (29) and (30) we obtain

\[
3^{x_2}(3^{x_3} - x_2) = q^{y_2}(q^{y_3} - y_2 + 1).
\]

If \( q \equiv \pm 1 \mod 9 \) then (37), (28), and (30) give \( 3^{x_3} \equiv 3 \mod 9 \) which is impossible. So we can apply Lemma 1 of [15] to (41) to see that

\[
3^{x_2} - 1 \mid y_3 - y_2.
\]

Now if \( 3^{x_2} < c/2 \), then \( q^{y_2} > c/2 > q^{y_3} / 2 \), contradicting (40), so we can assume

\[
3^{x_2} > c/2.
\]
So now, using (42) and (43) and letting \( k \geq 1 \) be some real number, (39) becomes
\[
\frac{3^{x_2-1}}{\log(3)} < \frac{2(\log(2) + x_2 \log(3))}{\log(3) \log(q)} + 22.997(\log(k) + (x_2 - 1) \log(3) - \log\log(3) + 2.405)^2. \tag{44}
\]
If (44) holds for some fixed \( x_2 \), then it also holds for that \( x_2 \) taking \( k = 1 \). So (44), combined with (38), gives \( x_2 \leq 7 \) (recalling \( x_2 \) odd). Now
\[
q^2 - q \leq q^{y_1} - q^{y_2} = 3^{x_2} - 1 \leq 2186, \tag{45}
\]
so that \( q \leq 47 \). We have already shown \( q \equiv 7 \mod 8 \) and \( q \equiv 2 \mod 3 \). So \( q = 23 \) or \( 47 \), both of which make (45) impossible.

**Case 3**

\[
p^{x_1} + (-1)^v = c \tag{46}
\]
\[
p^{x_2} + q^{y_2} = c \tag{47}
\]
\[
q^{y_3} + c = p^{x_3}, \tag{48}
\]
where \( v \in \{0, 1\} \) and \( p \) and \( q \) are odd primes. Consider first \( v = 1 \). Substituting (46) into (47) and (48) we find \( q^{y_2} \equiv -1 \mod p \) and \( q^{y_3} \equiv 1 \mod p \) so that \( 2 \mid y_2 \), which, by Lemma 1, is possible only when \((p, q, c) = (3, 5, 2)\) or \((5, 11, 4)\), both of which cases are impossible since \( c \leq 4 \) makes (47) impossible.

Now consider \( v = 0 \). Substituting (46) into (47) and (48) we get \( q^{y_2} \equiv 1 \mod p \) and \( q^{y_3} \equiv -1 \mod p \) so that \( 2 \mid y_2 \), which, by Lemma 1, is possible only when \((p, q, c) = (3, 5, 28)\) or \((5, 11, 126)\). \((p, q, c) = (3, 5, 28)\) makes (48) modulo 8 incompatible with (48) modulo 5, while \((p, q, c) = (5, 11, 126)\) makes (48) modulo 8 incompatible with (48) modulo 3.

**Case 4**

\[
2^{y_1} + (-1)^u = c \tag{49}
\]
\[
p^{x_2} + 2^{y_2} = c \tag{50}
\]
\[
2^{y_3} + c = p^{x_3}, \tag{51}
\]
where \( u \in \{0, 1\} \). From (49) and (50) we see that \( y_1 \geq 3 \) unless \((p, q, c) = (3, 2, 5)\), an excluded case. Clearly \( y_1 > y_2 \) and \( x_3 > x_2 \), so that Lemma 2.11 of [17] gives
\[
y_2 < y_1 < y_3, \tag{52}
\]
noting that the relevant exceptional cases of Lemma 2.11 of [17] have already been excluded.

Consider first \( u = 1 \). Substituting (49) into (50) and (51) and using (52), we find
\[
v_2(p^{x_2} + 1) = y_2 < y_1 = v_2(p^{x_3} + 1), \tag{53}
\]
so that \( p \equiv 3 \mod 4 \), \( x_3 \) is odd, and \( x_2 \) is even. But this makes (50) impossible modulo 8 since \( c \equiv 7 \mod 8 \) (recall \( y_1 \geq 3 \)).

Now consider \( u = 0 \). Substituting (49) into (50) and (51) and using (52), we find that
\[
v_2(p^{x_2} - 1) = y_2 < y_1 = v_2(p^{x_3} - 1)
\]
so that $2 \mid x_3$ which is impossible by Lemma 3 unless $(p, q, c) = (7, 2, 17)$, $(23, 2, 17)$, or $(2^t + 1, 2, 2^{t+1} + 1)$ where $t \geq 3$ (recall (22)). The first two of these three cases make (50) impossible, while the third case is the already excluded $(p, q, c) = (F, 2, 2F - 1)$.

**Case 5**

\[
\begin{align*}
2^{x_1} + (-1)^v &= c \quad (54) \\
2^{x_2} + q^{y_2} &= c \quad (55) \\
q^{y_3} + c &= 2^{x_3} \quad (56)
\end{align*}
\]

where $v \in \{0, 1\}$. We see that $x_2 < x_1 < x_3$. Also, $x_1 \geq 3$, otherwise (55) is impossible except when $(p, q, c) = (2, 3, 5)$, which has been excluded. Assume first $v = 1$. Then from (54) we get $c \equiv 7 \mod 8$. If $y_3$ is odd, then, from (56) we get $q \equiv 1 \mod 8$ so that (55) becomes impossible modulo 8. So $2 \mid y_3$ so that, using Lemma 2 and recalling (22), we see from (56) that we must have $(p, q, c) = (2, 11, 7)$, $(2, 181, 7)$, or $(2, 2^t - 1, 2^{t+1} - 1)$ where $t \geq 3$. The first two of these possibilities have $c = 7$, making (55) impossible, and the third possibility corresponds to the exceptional case $(2, M, 2M + 1)$ which we have already excluded.

So now assume $v = 0$. Substituting (54) into (55) and (56) we find that

\[
v_2(q^{y_2} - 1) = x_2 < x_1 = v_2(q^{y_3} + 1),
\]

which is possible only when $x_2 = 1$, so that $q = 2^{x_1} - 1$ and $c = 2^{x_1} + 1$, giving the exceptional case $(2, M, M + 2)$, which has been excluded.

**Case 6**

\[
\begin{align*}
p^{x_1} + 1 &= c \quad (57) \\
q^{y_2} + c &= p^{x_2} \quad (58) \\
q^{y_3} + c &= p^{x_3} \quad (59)
\end{align*}
\]

By Theorem 4 of [13], $p > 2$. Substituting (57) into (58) and (59) we find $q^{y_2} \equiv q^{y_3} \equiv -1 \mod p$, so that $2 \mid y_2 - y_3$, contradicting Theorem 3 of [13].

**Case 7**

\[
\begin{align*}
q^{y_1} + 1 &= c \quad (60) \\
q^{y_2} + c &= p^{x_2} \quad (61) \\
q^{y_3} + c &= p^{x_3} \quad (62)
\end{align*}
\]

By Theorems 3 and 4 of [13], $p > 2$ and $2 \nmid y_2 - y_3$. If $2 \mid x_2 - x_3$, then $p^{x_2} \equiv p^{x_3} \mod 3$ and $p^{x_2} \equiv p^{x_3} \mod 4$, so that

\[
q^{y_2} = p^{x_2} - c \equiv p^{x_3} - c = q^{y_3} \mod 12,
\]

so that $q \equiv 1 \mod 12$, $c \equiv 2 \mod 12$, and (61) gives $p = 3$, contradicting Corollary 1.7 of [2].
So we must have $2 \not| x_2 - x_3$. Without loss of generality take $x_2$ even and $x_3$ odd. Assume first $q > 2$. Then from (61) we see that $q^{y_2} + q^{y_1} + 1$ is a square, impossible by Lemma 1. So $q = 2$, and we can use equations (2), (4), and (6a) of [14] to see that $2^{y_2} \mid p - 1$. Now rewrite (61) as

$$2^{y_2} + (c - 1) = (p^{x_2} - 1)$$

to see that we must have $y_1 = y_2$, making the left side of (61) less than $2p$, which is impossible.

**Case 8**

$$p^{x_1} + 1 = c \quad (63)$$

$$q^{y_2} + c = p^{x_2} \quad (64)$$

$$p^{x_3} + c = q^{y_3} \quad (65)$$

Assume first $p > 2$. Substituting (63) into (64) and (65) we find $q^{y_2} \equiv -1 \mod p$ and $q^{y_3} \equiv 1 \mod p$, so that $2 \mid y_3$, contradicting Lemma 1.

So $p = 2$. Assume first $x_1 = 1$ so that $c = 3$. Then $q^{y_2} \equiv 5 \mod 8$, so that considering (65) modulo 8 we get $2 \not| y_3$, $x_3 = 1$, $(p, q, c) = (2, 5, 3)$, an excluded case. Assume next $x_1 = 2$ so that $c = 5$. If $q = 3$, we have the excluded case $(p, q, c) = (2, 3, 5)$, so we can assume $q > 3$. Considering (64) and (65) modulo 3 we get $q^{y_2} \equiv 2$, $q^{y_3} \equiv 1 \mod 3$, $2 \mid y_3$, $q^{y_3} \equiv 1 \mod 8$, $x_3 = 2$, $q = 3$, a contradiction. So $x_1 > 2$. (64) requires $q \equiv 7 \mod 8$ with $\left(\frac{p}{q}\right) = 1$; but then (65) gives $\left(\frac{p}{q}\right) = -1$, a contradiction.

**Case 9**

$$p^{x_1} + (-1)^w = c \quad (66)$$

$$p^{x_2} + q^{y_2} = c \quad (67)$$

$$p^{x_3} + q^{y_3} = c \quad (68)$$

where $w \in \{0, 1\}$. This case can be handled using essentially the same method as used to handle the case (31) in Theorem 7 of [14].

**Case 10**

$$1 + 1 = 2 \quad (69)$$

$$q^{y_2} + 2 = p^{x_2} \quad (70)$$

$$q^{y_3} + 2 = p^{x_3} \quad (71)$$

By Theorem 6 of [14] we cannot have both (70) and (71).

**Case 11**

$$1 + 1 = 2 \quad (72)$$

$$q^{y_2} + 2 = p^{x_2} \quad (73)$$

$$p^{x_3} + 2 = q^{y_3} \quad (74)$$
First suppose \( p \equiv q \equiv 7 \mod 8 \). Then (73) and (74) give \( \left( \frac{q}{p} \right) = \left( \frac{p}{q} \right) = -1 \), impossible when \( p \equiv q \equiv 3 \mod 4 \).

Now consideration modulo 8 with consideration modulo 3 shows that one of (73) or (74) must be of the form \( x^2 + 2 = 3^n \) for some integers \( x > 1 \) and \( n > 1 \); by Lemma 4.2 of [19] the only possibility is \( (p, q, c) = (3, 5, 2) \) or \( (5, 3, 2) \) which has been excluded.

Now we consider cases of three or more solutions to (3) with at least two solutions in which \( \min(x, y) = 0 \). Clearly there are at most two solutions with \( \min(x, y) = 0 \). Take \( \delta \in \{0, 1\} \). If \( \min(p, q) = 2 \), then \( c \) is odd so that the only possibility allowing two solutions with \( \min(x, y) = 0 \) is \( c = 3 \), and we have

\[
2 + 1 = 3, 2^2 - 1 = 3, -2^x + q^y = (-1)^\delta 3. \tag{75}
\]

If \( c = 2 \) we have the possibility of the following three solutions:

\[
1 + 1 = 2, 3 - 1 = 2, -3^x + q^y = (-1)^\delta 2. \tag{76}
\]

If \( y_3 = 1 \) in either (75) or (76) we have one of the last two exceptional cases in the formulation of Theorem 3; these have been excluded, so we can assume \( y_3 > 1 \) in both (75) and (76). By Lemma 2 of [14] we can assume \( \delta = 1 \) in both (75) and (76).

Now in (75) \( x_3 > 2 \) and consideration modulo 8 gives \( y_3 \) odd. So taking \( w = z = 1 \), we have

\[
(-q)^{y_3} + 2^{x_3}w^{y_3} = 3z^2, \]

from which we find that \( y_3 \) has no prime factor greater than or equal to 7 by Theorem 1.2 of [3]. Taking \( g \in \{3, 5\} \), we are left with the Thue equations

\[
x^g - 2^k y^g = -3,
\]

where \( 0 < k < g \) is chosen so that \( x_1 \equiv k \mod g \) (the case \( k = 0 \) is clearly impossible); the solutions to these Thue equations can be found using the PARI/GP command thue (see [11]), yielding only the single relevant case \( (p, q, c) = (2, 5, 3) \), which has been excluded. So we can assume we do not have \( y_3 > 1 \) in (75).

Now consider (76) with \( y_3 > 1 \) and \( \delta = 1 \). Lemma 4.2 of [19] shows that \( y_3 \) is odd (recall (22)), so that, taking \( w = z = 1 \) we have

\[
(-q)^{y_3} + 3^{x_3}w^{y_3} = 2z^3,
\]

from which we find that \( y_3 \) has no prime factor greater than 3 by Theorem 1.5 of [4]. So \( 3 \mid y_3 \), so that, considering (76) modulo 9, we get \( x_3 < 2 \), impossible. So we can assume we do not have \( y_3 > 1 \) in (76).

Now we can exclude (75) and (76) from consideration. So, in considering cases of three or more solutions to (3) with at least two solutions in which \( \min(x, y) = 0 \), we can assume that \( \min(p, q) > 2 \) and also that no solution has \( x = y = 0 \). Thus it remains to consider

\[
p^{x_1} = c + (-1)^w
\]

\[
q^{y_2} = c - (-1)^w
\]

\[
(-1)^w p^{x_3} + (-1)^w q^{y_3} = c
\]

where \( \min(x_1, y_2, x_3, y_3) > 0 \), \( u, v, w \in \{0, 1\} \), and \( \min(p, q) > 2 \). If \( (u, v) = (0, 0) \), then

\[
\frac{c + (-1)^w}{p} + \frac{c - (-1)^w}{q} = p^{x_1-1} + q^{y_2-1} \geq p^{x_3} + q^{y_3} = c,
\]

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impossible when \( \min(p, q) > 2 \). So it suffices to consider only the two cases given below by (77), (78), (79), and (83), (84), (85).

**Case 12**

\[
\begin{align*}
p^{x_1} + 1 &= c \\
1 + c &= q^{y_2} \\
q^{y_3} + c &= p^{x_3}
\end{align*}
\]

where \( p \) and \( q \) are odd primes.

From (77) and (78) we have

\[
\left( \frac{c}{p} \right) = 1 \quad (80)
\]

and

\[
\left( \frac{c}{q} \right) = \left( \frac{-1}{q} \right). \quad (81)
\]

From (77) and (78) we see that \( p \) and \( q \) cannot both be congruent to 1 mod 4. Considering the remaining possibilities for \( p \) and \( q \) modulo 4, we see that (80) and (81) are incompatible with (79) when \( 2 \nmid x_3y_3 \). Assume \( 2 \mid y_3 \). Then combining (80) and (79) we see that \( p \equiv 1 \mod 4 \), so that (77) gives \( c \equiv 2 \mod 4 \) while (79) gives \( c \equiv 0 \mod 4 \). So we are left with \( 2 \mid x_3 \) and \( 2 \nmid y_3 \). From (81) and (79) we have \( q \equiv 1 \mod 4 \), so that \( c \equiv 0 \mod 4 \) and, from (77), \( p \equiv 3 \mod 4 \) with \( x_1 \) odd. If \( 2 \mid y_2 \), then, since \( 2 \nmid x_1, 2 \mid x_3, \) and \( 2 \nmid y_3 \), we have

\[
v_2(c) = v_2(q^{y_2} - 1) > v_2(q^{y_2} - 1) = v_2(p^{x_3} - 1) > v_2(p^{x_1} + 1) = v_2(c),
\]

a contradiction. So we have

\[
2 \nmid x_1, 2 \nmid y_2, 2 \mid x_3, 2 \nmid y_3. \quad (82)
\]

If \( 3 \nmid pq \), then \( 3 \mid c \) and \( p \equiv 2 \mod 3 \). So now we have \( p \equiv 11 \mod 12 \) so that

\[
\frac{p - 1}{2} \equiv 5 \mod 6
\]

and there must be an odd prime \( r \) dividing \( p - 1 \) such that \( r \equiv 2 \mod 3 \). We have \( p^{x_1} \equiv p^{x_3} \equiv 1 \mod r \), \( c \equiv 2 \mod r \), \( q^{y_2} \equiv 3 \mod r \), \( q^{y_3} \equiv -1 \mod r \). But since \( 2 \mid y_3 - y_2 \), we must have

\[
\left( \frac{3}{r} \right) = \left( \frac{-1}{r} \right),
\]

which is impossible when \( r \equiv 2 \mod 3 \).

So \( 3 \mid pq \) and, recalling \( q \equiv 1 \mod 4 \), we are left with \( p = 3 \). We recall (82) and consider (77), (78), and (79) modulo 5. \( p^{x_1} \equiv \pm 2 \mod 5 \). If \( p^{x_1} \equiv 3 \mod 5 \) then, using (77) and (78), we get \( 3^{x_1} + 2 = 5^{y_2} \) so that Theorem 3 of [13] gives \( x_1 = y_2 = 1, c = 4, \) which has been excluded. So \( p^{x_1} \equiv 2 \mod 5, c \equiv 3 \mod 5, q^{y_2} \equiv q^{y_3} \equiv 4 \mod 5 \), so that (79) requires \( p^{x_3} \equiv 2 \mod 5 \), contradicting \( 2 \mid x_3 \) as in (82).

**Case 13**

\[
1 + c = p^{x_1} \quad (83)
\]
\[ q^{y_2} + 1 = c \quad (84) \]
\[ q^{y_3} + c = p^{x_3} \quad (85) \]

where \( p \) and \( q \) are odd primes.

Substituting (83) into (85) and applying Lemma 1 we find that we can assume \( y_1 \) is odd, since the exceptional cases of Lemma 1 make (84) impossible since \( c \leq 4 \) and \( q \geq 5 \). Substituting (84) into (85) and applying Lemma 1, we find that we can assume \( x_3 \) is odd. So

\[ 2 \not| x_3, 2 \not| y_3. \quad (86) \]

We have

\[ \left( \frac{c}{q} \right) = 1 \quad (87) \]

and

\[ \left( \frac{c}{p} \right) = \left( \frac{-1}{p} \right). \quad (88) \]

If \( 2 \mid x_1 \), we have \( 4 \mid c, q \equiv 3 \mod 4 \), and, from (85) and (86), \( p \equiv 3 \mod 4 \). Combining (87) with (85) we get \( \left( \frac{q}{p} \right) = 1 \), while combining (88) with (85) we get \( \left( \frac{c}{p} \right) = 1 \), which is impossible when \( p \equiv q \equiv 3 \mod 4 \). So \( 2 \not| x_1 \).

Therefore, if \( 3 \mid c \), (83) gives \( p \equiv 1 \mod 3 \). But (84) gives \( q \equiv 2 \mod 3 \), and, from (85) and (86), we have a contradiction. So \( 3 \not| c \).

So \( 3 \mid pq \). If \( q = 3 \) then, from (84), we get \( c \equiv 1 \mod 3 \), and, from (83) we get \( p \equiv 2 \mod 3 \). But then (85) requires \( 2 \mid x_3 \), contradicting (86). So \( p = 3 \).

To handle the case \( p = 3 \), we use Lemma 4 with the following substitutions: \( a = 3, b = q, x = x_3, y = y_3 \).

Then by Lemma 4 (noting \( c > 1 \)) we must have either (38) or (39). Combining (83) and (85) we get

\[ 3^{x_1}(3^{x_3-x_1} - 1) = q^{y_3} - 1 \]

so that

\[ 3^{x_1} \mid q^{y_3} - 1. \quad (89) \]

From (83) and (84) we get \( x_1 > 1 \), so that \( q^{y_2} \equiv 7 \mod 9, q \not\equiv \pm 1 \mod 9 \). Applying Lemma 1 of [15] to (89), we have

\[ 3^{x_1-1} \mid y_3. \quad (90) \]

Using (90) and (83) and noting that if (21) holds for \( G = G_1 > G_0 > 1 \) it holds for \( G = G_0 \), we see that (38) and (39) can be replaced by

\[ \frac{3^{x_1-1}}{\log(3)} < 2409.08 \quad (91) \]

and

\[ \frac{3^{x_1-1}}{\log(3)} < \frac{2x_1}{\log(q)} + 22.997((x_1 - 1) \log 3 - \log \log 3 + 2.405)^2, \quad (92) \]

giving \( x_1 \leq 8 \). Using (83), (84), (85), and (86) we have

\[ 3^{x_1} - 2 = q^{y_2}, 3^{x_3} \equiv 1 \mod q, 2 \not| x_3. \quad (93) \]

We easily check that (93) is impossible for \( x_1 = 3, 5, \) or \( 7 \) (recall \( 2 \not| x_1 > 1 \)).
We have now shown that the list of exceptional cases in Theorem 3 includes all \((p, q, c)\) allowing at least three solutions to (3). It remains to show that for each such \((p, q, c)\) the list of solutions \((x, y)\) is complete.

Consider first \((p, q, c) = (2, 2^t + (-1)^3, 3)\) which gives the three solutions

\[2 + 1 = 2^2 - 1 = -(-1)^3 2x_3 + (-1)^3 q^{y_3} = 3,\]

where \(y_3 = 1\) and \(x_3 > 1\). If \(q = 2^t + 3\), then we cannot have \(q^{y_4} + 3 = 2^{x_4}\) since \(q \equiv 3 \mod 4\). So any further solution \((x_4, y_4)\) must be of the form \(2^{x_4} + 3 = q^{y_4}\) with \(y_4\) odd so that \(q^{y_4} \equiv q^{y_3} \mod 3\), giving \(2 \mid x_3 - x_4\), contradicting Theorem 3 of [13], so that there exactly three solutions in this case. Similarly, the case \(q = 2^t - 3\) gives exactly three solutions (note that \(t\) is defined so that \(q \neq 5\)).

Now consider \((p, q, c) = (3, 3^n + (-1)^2, 2)\) which gives the three solutions

\[1 + 1 = 3 - 1 = -(-1)^3 3x_3 + (-1)^3 q^{y_3} = 2.\]

By the results given in Cases 10 and 11, this case also has exactly three solutions (except for the excluded case \((3, 5, 2)\)).

The remaining cases can be handled either by Theorem 2 of [13] or by Observation 8 of [15].

The proof of Theorem 3 is elementary except for the use of Lemma 4 (to handle the case \(p = 3\) in Cases 2 and 13), Corollary 1.7 of [2] (to handle the case \(p = 3\) in Case 7), Lemma 3 (Case 4), Lemma 2.11 of [17] (Cases 2 and 4), Observation 8 of [15] (at the end of the proof of Theorem 3), and, finally, Theorem 1.2 of [3], Theorem 1.5 of [4], and Pari (to obtain \(y_3 = 1\) in (75) and (76)). The following Lemma 5 allows us to replace Observation 8 of [15] by an elementary result, and the Corollary to Lemma 5 shows that Lemma 2.11 of [17] can be given an elementary proof; also the somewhat longer alternate proof of Case 4 of Theorem 3 given below removes the dependence on Lemma 3, thus removing the dependence on [1]. Finally, rewriting the last two exceptional \((p, q, c)\) in the formulation of Theorem 3 as \((3, (3^n + (-1)^2)^{1/m}, 2)\) and \((2, (2^t + (-1)^3)^{1/m}, 3)\) where \(m \geq 1\) is an integer, we can remove the need for Theorem 1.2 of [3], Theorem 1.5 of [4], and Pari. With these changes the proof of Theorem 3 is lengthened but becomes elementary except for three applications (all with \(\min(p, q) = 3\)) of lower bounds on linear forms in logarithms (note that Corollary 1.7 of [2] and Lemma 4 both use a theorem of Mignotte [10] as used in [2]).

**Lemma 5.** Redefine (3) to allow \(q\) to be composite. Then:

If \((p, q, c) = (2, M, M + 2)\) where \(M = 2^t - 1 > 3\), the only solutions to (3) are

\[2^t + 1 = c, \quad 2 + M = c, \quad 2^{t+1} - M = c.\]  

If \((p, q, c) = (2, M, 2M + 1)\) where \(M = 2^t - 1 > 3\), the only solutions to (3) are

\[2^{t+1} - 1 = c, \quad 2^t + M = c, \quad 2^{2t} - M^2 = c.\]

If \((p, q, c) = (2, F, F - 2)\) where \(F = 2^t + 1 > 5\), the only solutions to (3) are

\[2^t - 1 = c.\]
Alternate Proof of Case 4 of Theorem 3.

Corollary to Lemma 5.

requires \( y \) of [15] can be replaced by the use of either Theorem 2 of [13] or Lemma 5 above. But in every case the use of Theorems 1 and 7 of [15].

The proof of Lemma 2.11 of [17] depends only on the lemmas preceding it in that paper, which in turn are elementary except for use of Theorems 1 and 7 of [15]. But in every case the use of Theorems 1 and 7 of [15] can be replaced by the use of either Theorem 2 of [13] or Lemma 5 above.

Alternate Proof of Case 4 of Theorem 3. It suffices to treat only the case \( u = 0 \), noting \( y_1 \geq 3 \) and recalling \( 2 \mid x_3 \). If \( p \equiv 7 \mod 8 \) then (50) requires \( \left( \frac{c}{M} \right) = 1 \) while (51) requires \( \left( \frac{c}{M} \right) = -1 \), so

\[ p \not\equiv 7 \mod 8. \] (106)

If \( y_2 = 1 \) then \( p^{y_2} = 2^{y_2} - 1 \equiv 7 \mod 8, \) impossible by (106), so \( p^{y_2} \equiv 1 \mod 4. \) If \( 2 \mid x_2 \) then, using Lemma 2 with (22) and (106), we must have \( (p, c) = (11, 129) \) or \( (181, 32769) \), so considering (51) modulo 5 we find \( 2 \nmid y_3 \), while considering (51) modulo 3 we find \( 2 \mid y_3 \) since \( 2 \mid x_3 \). So

\[ 2 \nmid x_2, \] (107)
and
\[ p \equiv 1 \mod 4. \]  
(108)

Assume now \( 4 \mid x_3 \) and recall (22). Then, since
\[ 2^{y_3} + 2^{y_1} + 1 = p^{x_3}, \]  
(109)

consideration modulo 5 gives \( 2^{y_3} + 2^{y_1} \equiv 0 \mod 5 \), so that \( 2 \mid y_3 - y_1 \). But consideration of (109) modulo 3 gives \( 2 \not\mid y_3 - y_1 \), a contradiction. So
\[ 2 \mid| x_3. \]  
(110)

Let \( k = v_2(p - 1) \). Then, using (107) and (110), we have
\[ v_2(p^{x_2} - 1) = v_2(p^{x_3/2} - 1) = k. \]  
(111)

From (111) and (108) we have \( v_2(p^{x_3} - 1) = k + 1 \), so from (109) and (52) we have \( y_1 = k + 1 \), so from (50) and (111) we have \( y_2 = k \), \( p = 2^k + 1 \) (note \( x_2 = 1 \) by (22)), giving the already excluded case \((p, q, c) = (F, 2, 2F - 1)\).

\[ \square \]

References


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