# HECKE THEORY OVER COMPLEX QUADRATIC FIELDS

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## ABSTRACT

This thesis generalizes work of Stark and Patterson-Goldfeld to complex quadratic fields of class number greater than one. We first define a vector Hecke operator which has a classical-looking multiplicative relation. We then define vector forms on the quaternionic upper half space, and examine their Fourier expansions and the effect of the Hecke operators on these coefficients. Then we examine eigenforms of the Hecke operators, and Dirichlet series formed out of these vector forms. One obtains both Euler products and functional equations for the Dirichlet series. Finally, we define quaternionic Eisenstein series and study their Fourier expansions and Dirichlet series. We obtain results that are remarkably close to the classical analogs.

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## ACKNOWLEDGMENTS

"A student is not above his teacher, but everyone who is fully trained will be like his teacher" (Luke 6.40, NIV). God has surely blessed me with tremendous teachers. Miss Slater interested me in math in seventh grade, Ruth Delcamp encouraged me in ninth, Prof. Sigler in twelfth, Prof. Shatz at Penn, and now Prof. Stark for the last few years. I also owe many debts to Prof. Ankeny and Prof. Goldfeld for teaching me a large part of what little I know of number theory. I am also grateful for colleagues and friends who have taught me a great deal not only about number theory but also about life. Special thanks belong to Ram Murty, Bob Rumely, Prof. Patterson, Prof. Terras, Sol Friedberg, and Jon Sands. Again, I want to specifically acknowledge Dorian Goldfeld and Harold Stark as the chief brain trusts behind this thesis, and to thank Prof. Stark for his continual patience and generosity during the last two years.

## I. Introduction

From the time of Gauss and Ramanujan, people have been interested in the coefficients of modular forms. One of the most useful tools used to study these coefficients is the Hecke algebra. Hecke's theory enables one to see the connections between the modular forms and the associated Dirichlet series.

The classical modular forms were quickly generalized to Hilbert modular forms, defined over totally real fields. Whenever the class number of the field was greater than one, however, people found obstacles to extending Hecke theory to Hilbert modular forms. Oskar Hermann [5] finally overcame these difficulties by introducing the idea of a vector of modular forms, each component of the vector form corresponding to an ideal class of the field. He uses ideal numbers to essentially introduce a principal generator for each ideal. Eichler [1] has defined vector forms without using ideal numbers, but only by replacing the ideal numbers by complicated conditions on the ideals involved.

Ideal numbers are very appealing; essentially their origins go back to Kummer. Hecke [4] developed them fully in 1920 in order to replace ideals by specific elements. Ideal numbers behave like algebraic integers except for an important restriction on addition. One can only add ideal numbers within a given class, and this forces one to consider classes of matrices and classes of forms. This is why Hermann introduces vector forms and vector Hecke operators.

Hilbert modular forms are only defined over totally real fields. Recently, several people have considered the complex field analogs for modular forms. One introduces the quaternionic upper half space which supports an action by $SL(2, \mathbb{C})$. One also introduces representations of the quaternions in order to guarantee the needed transformation properties. For fields of class number one, Patterson-Goldfeld [8] have developed Eisenstein series defined on the quaternionic upper half space. Stark [10] has extended Hecke operators to the complex case if the field has class number one.

This thesis extends the work of Patterson-Goldfeld and of Stark to complex quadratic fields of class number greater than one. The concepts used parallel those of Hermann, so we also introduce vector forms and vector Hecke operators. In the complex case, however, even when the class number is one, the forms are vectors, so we will actually need to consider vectors of vectors, which makes the notation slightly awkward. Nevertheless, the basic results parallel the classical cases. In Chapter II we introduce ideal numbers and develop the needed properties. One can then develop the Hecke operators, which look remarkably similar to the classical case. Chapter III begins with a short discussion of the quaternionic upper half space and of quaternionic representations. We now have all of the preliminaries.

Now one can develop the theory of modular forms. As mentioned before, we actually have a vector of forms, one component for each ideal class. These components in turn are vectors, each a modular form with respect to a fixed class of matrix operators. One can introduce Fourier expansions in a very general setting — the example of the Eisenstein series in Chapter IV shows the reasonableness of our definition. After suitably renormalizing the Hecke operators, one considers simultaneous eigenforms and finds analogs to many of the classical results. For instance, the Fourier coefficients correspond closely to the eigenvalues of the Hecke operators. They are not equal, however, which would be impossible since the Fourier coefficients actually are a vector themselves, while the eigenvalues are scalars. Nevertheless, the correspondence is very close, and one can show that the multiplicativity of the coefficients is related to the vector form being an eigenform for the Hecke operators.

Dirichlet series are found via two techniques. The Mellin transform defines a Dirichlet series which has a functional equation. The eigenvalues of the Hecke operators define another Dirichlet series which has an Euler product. These Dirichlet series are not the same, however, since the Fourier coefficients are not exactly equal to the eigenvalues. One suitably twists the series, therefore, to create a new Dirichlet series which has both an Euler product and a functional equation.

After wrestling with the complications of vectors of forms and vector Hecke operators, one could wish that the principal component corresponding to the principal ideal class might contain enough information to reconstruct the entire form. If so, then one would be able to eliminate ideal numbers and the complicated vectors of vectors. The principal theorem in Chapter III Section 8 shows that one's wish can be fulfilled. If a function $f$ is an eigenform for "principal" Hecke operators and is modular with respect to the "principal" matrices, then one can find a vector eigenform which has $f$ as its principal component.

The final chapter illustrates the results of Chapter III by defining quaternionic Eisenstein series. We explicitly find the

Fourier coefficients and the Hecke eigenvalues, which are remarkably similar to the results one gets in the classical cases. We explicitly evaluate the matrix integral which appears in the Fourier expansion, and show that one gets a matrix with sums of Bessel functions. This is slightly surprising, but a consideration of the differential equations satisfied by the integrals confirms that one gets Bessel functions. In case the weight is one, a quaternion-valued Eisenstein series can be defined for which an interesting differential equation can be found, but for other weights this equation does not seem to have an obvious analogy. Finally, we develop an example of Eisenstein series over $K = Q(\sqrt{-23})$.

In conclusion, the complications of class number greater than one are relatively minor. The broad outlines are precisely those found in the classical analogs. Furthermore, since the principal components determine the entire form, one need not even consider ideal numbers nor a vector of vectors, with a separate component for each ideal class.

## II. 1 Ideal Numbers

Ideal numbers are an appealing concept, underlying the original ideas of Kummer. Hecke [ 4 ] developed a full theory of ideal numbers in 1920, which is summarized in Hermann [ 5 ]. In this section I will establish the notation and basic properties of ideal numbers.

Let $K$ be an algebraic number field, and let $O(K)$ be the algebraic integers of $K$, and let $U$ be the units of $O(K)$. Let $\underline{I}$ be the group of ideals over $O(K)$, and $\underline{P}$ the subgroup of principal ideals. Then classical algebraic number theory shows that $\underline{I}/\underline{P}$ is a finite abelian group of order $h$, called the ideal class number. We can decompose $\underline{I}/\underline{P}$ into a direct product of cyclic subgroups, say $G_1 \times G_2 \times \cdots \times G_r$. Let $N_s = \text{order } (G_s)$, $s = 1, 2, \ldots, r$.

For any ideal $\underline{a}$, let $[\underline{a}]$ denote the class of $\underline{a}$ in $\underline{I}/\underline{P}$. Choose ideals $\underline{b}_s$ so that $[\underline{b}_s]$ generates $G_s$, $s = 1, 2, \ldots, r$. Then $\underline{b}_s^{N_s}$ is a principal ideal, so let $\underline{b}_s^{N_s} = c_s \cdot O(K) = (c_s)$ for some $c_s \in K$. Let $\beta_s = \sqrt[N_s]{c_s}$ where we fix some $N^{th}$ root of $c_s$, with $N = \ell.c.m\,(N_1, N_2, \ldots, N_r)$.

Suppose $\underline{a}$ is any ideal. Then $\underline{a} = (c)\underline{b}_1^{P_1} \cdot \underline{b}_2^{P_2} \cdots b_r^{P_r}$ for some integer powers $P_s$ mod $N_s$ and for some $c \in K$. Define $\alpha = c \cdot \beta_1^{P_1} \cdots \beta_r^{P_r}$. Then $\alpha$ is said to be an ideal number associated to the ideal $\underline{a}$. Let $K[\underline{a}] = \{d\alpha \mid d \in K\}$.

Before proceeding, I must comment on the arbitrary choices we have made in defining $\alpha$. Firstly, the decomposition $\underline{I}/\underline{P} = G_1 \times G_2 \times \cdots \times G_r$ is not unique, nor the choice of generators $\underline{b}_s$. Secondly, the choices of $c_s$ and $c$ are only determined modulo U, and also the $N^{th}$ root is fixed arbitrarily. Fortunately, however, N is the same for any decomposition $G_1 \times \cdots \times G_r$. Changing the generators $\underline{b}_s$ changes $\alpha$ by a principal element (up to an arbitrary $N^{th}$ root of unity) so $K[\underline{a}]$ is unchanged. The arbitrary $N^{th}$ root is more critical. Changing the choice of $c_s$ generating $\underline{b}_s$ and changing the branch of the root will change $\alpha$ by an arbitrary $N^{th}$ root of a unit $u \in U$. In particular, for a complex quadratic field with $h > 1$, $U = \{1, -1\}$ so the ideal numbers are unique only up to an arbitrary $2N^{th}$ root of unity.

Notation. Small underlined Latin letters will denote ideals of $\underline{I}$. Small Greek letters will be reserved for ideal numbers. Propositions and theorems will be numbered, and in case they have several parts, a second number will denote the specific part. For instance, Prop. 4.3 means Proposition 4, part number 3.

II. 2 Properties of Ideal Numbers

Let $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_h$ be ideal class representatives. Define the set of ideal numbers to be $Z = \bigcup_{m=1}^{h} K[\underline{a}_m]$. If $[\underline{a}] = [\underline{a}_m]$ then $K[\underline{a}] = K[\underline{a}_m]$. Notice that if $[\underline{a}] \neq [\underline{a}_m]$ then $K[\underline{a}] \cap K[\underline{a}_m] = \{0\}$.

The multiplication of $K$ extends to $Z$, and in fact $K[\underline{a}] \cdot K[\underline{b}] = K[\underline{a} \cdot \underline{b}]$, so $Z$ is a multiplicative group. Given $\alpha \in Z$, we have $\alpha = c \cdot \beta_1^{P_1} \beta_2^{P_2} \cdots \beta_r^{P_r}$ for some $c \in K$, $P_s \bmod N_s$, so we can associate an ideal $\underline{a} = (c)\underline{b}_1^{P_1} \underline{b}_2^{P_2} \cdots \underline{b}_r^{P_r}$. This association is a surjective homomorphism $(-): Z \to \underline{I}$ given by $\alpha \to (\alpha) = \underline{a}$. The kernel is U.

For any fixed $[\underline{a}]$, the addition of $K$ extends to $K[\underline{a}]$, so $K[\underline{a}]$ is an additive group. One cannot, however, add two ideal numbers from different $K[\underline{a}]$. In other words, $\alpha + \beta$ is defined iff $[(\alpha)] = [(\beta)]$.

For convenience, define $[\alpha] = [(\alpha)]$ and $[\alpha\underline{a}] = [(\alpha)\underline{a}]$ for any $\alpha \in Z$ and $\underline{a} \in \underline{I}$. One has that $[\alpha]^{-1} = [\alpha^{-1}]$.

We define $\alpha \in Z$ to be an integral ideal number if $\alpha$ is an algebraic integer which happens iff $(\alpha)$ is an integral ideal. Let $O[\underline{a}]$ be the integral elements of $K[\underline{a}]$ and $O(Z) = \bigcup_{m=1}^{h} O[\underline{a}_m]$ be the integral elements of Z. If $\alpha, \beta \in Z$, we say that $\alpha$ divides

$\beta$, written $\alpha \mid \beta$, if $\beta/\alpha$ is an integral ideal number. Note that $\alpha | \beta$ in Z iff $(\alpha) | (\beta)$ in $\underline{I}$. From this observation immediately follow the multiplicative properties of greatest common divisors and least common multiples for ideal numbers. We can also define congruences: if $\alpha, \beta, \gamma \in Z$, we say $\alpha \equiv \beta \bmod \gamma$ when $(\alpha - \beta)/\gamma$ is integral. In particular, $[\alpha] = [\beta]$ so that $\alpha - \beta$ will be defined. The properties of congruences in K extend to Z. In particular, for $\alpha \in O(Z)$, define $N(\alpha) = N((\alpha))$ where $N((\alpha))$ is the norm of the ideal $(\alpha)$ which is also the number of incongruent representatives $\bmod (\alpha)$ in K. The following proposition summarizes useful facts we will need later.

Proposition 1. Define $(\alpha, \beta)$ = greatest common divisor $(\alpha, \beta)$.

1) Let $[\underline{c}]$ be any ideal class, $\alpha, \beta \in Z$, and $\alpha | \beta$. Then there exists $\gamma \in K[\underline{c}]$ so that $\alpha = (\beta, \gamma)$.

2) Let $\gamma = (\alpha, \beta)$. Then there exists $\mu, \nu \in O(Z)$ so that $\gamma = \alpha\mu + \beta\nu$.

3) Given $\alpha, \beta, \gamma \in Z$, $\alpha\mu \equiv \beta \bmod \gamma$ has an integral solution $\mu \in O(Z)$ iff $(\alpha, \gamma) | \beta$.

4) Let $[\underline{a}]$ be any ideal class, $\beta, \gamma \in O(Z)$. Then in $K[\underline{a}]$ there are $N(\beta)$ residue classes mod $\beta$, and furthermore we can choose representatives $\beta_1, \beta_2, \ldots, \beta_{N(\beta)} \in O[\underline{a}]$ so that $(\beta_n, \gamma) = 1$

for all $n = 1, 2, \ldots, N(\beta)$. Finally if $\delta \in O(Z)$ with $(\delta, \beta) = 1$, then $\delta\beta_1, \delta\beta_2, \ldots, \delta\beta_{N(\beta)}$ is a complete residue system mod $\beta$ in $K[\underline{a}\,\delta]$.

5) Let $(\alpha, \delta) = 1$ and suppose $\alpha\beta \equiv \alpha\gamma \bmod \delta$. Then $\beta \equiv \gamma \bmod \delta$.

6) Define $\pi \in O(Z)$ to be prime if $(\pi)$ is a prime ideal. Then for any $\alpha \in Z$, $\alpha = u\pi_1^{p_1}\pi_2^{p_2} \cdots \pi_k^{p_k}$ where $u \in U$, each $\pi_n$ is prime, and each $p_n$ is a rational integer.

<u>Proof</u>. The proofs for 1) and 6) follow from the corresponding properties for ideals. The proof of 2) is essentially the Euclidean algorithm, except that one must be careful to add numbers only when they lie in the same $K[\underline{a}]$. The crucial point that allows the algorithm to work is that zero is in $O[\underline{a}]$ for every $[\underline{a}]$. The proofs for 3) and 4) follow from the corresponding properties in $O(K)$, once one multiplies by an appropriate ideal number to put everything into the principal class. Then the proof of 5) follows since 3) implies that an inverse exists. <u>QED</u>

## II. 3 <u>Ideal Number Matrices</u>

Let $GL(2, \mathbb{C})$ denote all $2 \times 2$ invertible complex matrices. Let $[\underline{a}], [\underline{c}]$ be ideal classes, and $\mu \in O(Z)$. Define a set of matrices

$$\Gamma^{\underline{a}}_{\underline{c}}(\mu) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(2, \mathbb{C}) \;\middle|\; \alpha\delta - \beta\gamma = \mu , \right.$$

$$\left. \alpha \in O[\underline{a}],\ \gamma \in O[\underline{c}],\ \beta \in O[\mu\underline{c}^{-1}],\ \delta \in O[\mu\underline{a}^{-1}] \right\} .$$

Note that $\Gamma^{\underline{a}}_{\underline{c}}(\mu) = \Gamma^{\underline{b}}_{\underline{d}}(\mu)$ iff $[\underline{a}] = [\underline{b}]$ and $[\underline{c}] = [\underline{d}]$ .

For convenience we will write

$$\Gamma_{\underline{c}}(\mu) = \Gamma^{(1)}_{\underline{c}}(\mu) , \quad \Gamma^{\underline{a}}_{\underline{c}} = \Gamma^{\underline{a}}_{\underline{c}}(1), \text{ and } \Gamma_{\underline{c}} = \Gamma^{(1)}_{\underline{c}}(1) .$$

We now fix a representative of each $\Gamma^{\underline{a}_m}_{\underline{a}_n}$ .

Choose some $C^{\underline{a}_m}_{\underline{a}_n} \in \mathcal{C}^{\underline{a}_m}_{\underline{a}_n}$ for all $m, n = 1, 2, \ldots, h$. As usual, we let $C^{\underline{a}}_{\underline{b}} = C^{\underline{a}_m}_{\underline{a}_n}$ whenever $[\underline{a}] = [\underline{a}_m]$ and $[\underline{b}] = [\underline{a}_n]$. For convenience we may choose $C^{(1)}_{\underline{a}_m} = I$ for $m = 1, 2, \ldots, h$.

Proposition 2

1) (the consistency rule). Let $A \in \Gamma^{\underline{a}}_{\underline{c}}(\mu)$ and $B \in \Gamma^{\underline{b}}_{\underline{d}}(\nu)$. Then $A \cdot B$ is defined iff $[\underline{d}] = [\underline{a}\,\underline{b}\,\underline{c}\,\mu^{-1}]$. We have

$$\Gamma^{\underline{a}}_{\underline{c}}(\mu)\ \Gamma^{\underline{b}}_{\underline{a}\,\underline{b}\,\underline{c}\,\mu^{-1}}(\nu) = \Gamma^{\underline{a}\,\underline{b}}_{\underline{c}\,\underline{b}}(\mu\nu) .$$

2) $$\Gamma^{\underline{a}}_{\underline{c}}(\mu) \cdot C^{\underline{b}}_{\underline{a}\,\underline{b}\,\underline{c}\,\mu^{-1}} = \Gamma^{\underline{a}\,\underline{b}}_{\underline{c}\,\underline{b}}(\mu) .$$

3) $$C^{\underline{b}}_{\underline{c}\,\underline{a}^{-1}\underline{b}^{-1}}\,\Gamma^{\underline{a}}_{\underline{c}}(\mu) = \Gamma^{\underline{a}\,\underline{b}}_{\underline{c}\,\underline{b}^{-1}}(\mu)\ .$$

4) $$C^{\underline{b}}_{\underline{c}\,\underline{a}^{-1}\underline{b}^{-1}}\,C^{\underline{a}}_{\underline{c}} = A \cdot C^{\underline{a}\,\underline{b}}_{\underline{c}\,\underline{b}^{-1}} \text{ for some } A \in \Gamma_{\underline{a}^{-1}\underline{b}^{-2}\underline{c}}\,.$$

5) Given $A \in \Gamma_{\underline{a}\,\underline{c}}$ there is $A' \in \Gamma_{\underline{a}^{-1}\underline{c}}$ so that $C^{\underline{a}}_{\underline{c}}A = A'C^{\underline{a}}_{\underline{c}}$.

6) If $B \in \Gamma^{\underline{a}}_{\underline{c}}$ then $B^{-1} \in \Gamma^{\underline{a}^{-1}}_{\underline{c}}$.

Proof. These follow from straightforward matrix manipulation. QED

Remarks. It is impossible to overemphasize the importance of the consistency rule Prop. 2.1. We wish to develop Hecke theory, so we must require that the determinant of a matrix be defined. But as noted above, addition is only defined within a fixed $K[\underline{a}]$. Thus, one must take care that one multiplies matrices where the additions will be defined.

Later it will be convenient to identify matrices that differ only by elements of $\Gamma_{\underline{a}_m}$. We will say that $B \sim B'$ whenever there is an $A \in \Gamma_{\underline{a}_m}$ for some $m = 1, 2, \ldots, h$ so that $B = AB'$. We extend this to formal sums of matrices by saying that $B + D \sim B' + D'$ whenever $B \sim B'$ and $D \sim D'$.

II.4 Vectors of Matrix Operators

In the next chapter, I will define vector modular forms, and now I will define the vectors of matrices that will act on these forms. One should think of these vectors of matrices as operators, so in particular addition will be formal operator addition rather than matrix addition.

Let $\mu \in O(Z)$ and $[\underline{b}]$ be an ideal class. Choose representatives $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_h$ from each ideal class, with $\underline{a}_1 = (1)$ representing the principal class. This choice will remain fixed for the remainder of the thesis. Define a vector operator

$$A = A(\mu, \underline{b}) = \left(A_{[\underline{a}_m]}\right)_{m=1,2,\ldots,h} = \begin{pmatrix} A_{[\underline{a}_1]} \\ A_{[\underline{a}_2]} \\ \vdots \\ A_{[\underline{a}_h]} \end{pmatrix}$$

where $A_{[\underline{a}_m]} \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}_m}(\mu)$ for $m = 1, 2, \ldots, h$.

For the remainder of this chapter, we will be dealing with many vectors of length h. I will reserve the letter m to run through the set $1, 2, \ldots, h$. I will also drop the subscripting index from vectors, and one must infer from the context and from the undetermined m that a vector of length h is indicated. For instance, $\left(A_{[\underline{a}_m]}\right)_{m=1,2,\ldots,h}$ will be written $\left(A_{[\underline{a}_m]}\right)$.

Let $B = B(\mu, \underline{b}) = \left(B_{[\underline{a}_m]}\right)$ be another vector of matrices $B_{[\underline{a}_m]} \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}_m}(\mu)$. Then we define a formal addition by $A + B = \left(A_{[\underline{a}_m]} + B_{[\underline{a}_m]}\right)$, where the addition is formal addition of operators, not as matrices. We do not define addition of vectors $A(\mu, \underline{a})$ and $B(\nu, \underline{b})$, not even formally, unless $\mu = \nu$ and $[\underline{a}] = [\underline{b}]$.

Let $M(\mu, \underline{b})$ be the set of all formal sums of vector operators $A(\mu, \underline{b})$. Note that $[\underline{a}] = [\underline{b}]$ implies that $M(\mu, \underline{a}) = M(\mu, \underline{b})$. For convenience, we set $M = M(1, (1))$, $M(\mu, \underline{I}) = \bigcup_{m=1}^{h} M(\mu, \underline{a}_m)$, and $M(Z, \underline{I}) = \bigcup_{\mu \in Z} M(\mu, \underline{I})$. We can define multiplication in $M(Z, \underline{I})$ as follows:

$$A(\mu, \underline{a}) \cdot B(\nu, \underline{b}) = D(\mu\nu, \underline{a}\,\underline{b}) = \left(D_{[\underline{a}_m]}\right)$$

where

$$D_{[\underline{a}_m]} = A_{[\underline{a}_m]} \cdot B_{[\underline{a}^2 \underline{a}_m \mu^{-1}]} \quad .$$

In light of the consistency rule Prop. 2.1, this is legitimate.

Choose $C(\underline{b}) = \left(C^{\underline{b}}_{\underline{b}\,\underline{a}_m}\right)$ where we fix some choice of

$C^{\underline{b}}_{\underline{b}\,\underline{a}_m} \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}_m}$ for all $[\underline{b}]$ and $m = 1, 2, \ldots, h$. I will reserve the letter $C$ to be these fixed representatives throughout this chapter. For convenience, fix $C^{(1)}_{\underline{a}_m} = I$ for all $m$ so that

$$C((1)) = \begin{pmatrix} I \\ I \\ \vdots \\ I \end{pmatrix}$$ will be the multiplicative identity in $M(Z, \underline{I})$.

$M(Z, \underline{I})$ has the following properties.

<u>Proposition 3</u>

1) (Distributive Law). Let $A, B, D \in M(Z, \underline{I})$ and suppose $A + B$ is defined. Then

$$(A + B)D = AD + BD \quad \text{and}$$

$$D(A + B) = DA + DB \ .$$

2) $M(Z, I)$ and $M$ are multiplicative semigroups.

3) $M(\mu, \underline{b}) = C(\underline{b}) \cdot M(\mu, (1)) = M(\mu, (1)) \cdot C(\underline{b})$ .

4) $M(\mu, \underline{b}) = M \cdot M(\mu, \underline{b}) = M(\mu, \underline{b}) \cdot M$ .

5) $M \cdot C(\underline{b}) = C(\underline{b}) \cdot M$ .

6) $C(\underline{a})\, C(\underline{b}) = A\, C(\underline{a}\,\underline{b})$ for some $A \in M$ .

<u>Proof.</u> Everything follows from Prop. 2 along with the definition of multiplication. <u>QED</u>

When we define vector modular forms, they will be invariant under $M$. Hence it is convenient to define equivalence of operators. We say that $B \sim B'$ if there exists $A \in M$ so that $B = AB'$. In particular, $B$ and $B'$ are in the same $M(\mu, \underline{b})$. We extend this to formal sums by defining $B + D \sim B' + D'$ whenever $B \sim B'$ and $D \sim D'$. Notice that if

$B = \left(B_{[\underline{a}_m]}\right)$ and $D = \left(D_{[\underline{a}_m]}\right)$, then $B \sim D$ iff

$B_{[\underline{a}_m]} \sim D_{[\underline{a}_m]}$ for $m = 1, 2, \ldots, h$. Thus there should be no confusion between equivalence of vector operators and equivalence of matrices defined earlier.

Proposition 4

1) $\sim$ is an equivalence relation.

2) For any $D$, $B \sim B'$ implies $BD \sim B'D$.

3) Suppose that $D \in M(Z, \underline{I})$ satisfies $D \sim DA$ for all $A \in M$. Then $B \sim B'$ implies $DB \sim DB'$.

Conversely, suppose that $B \sim B'$ implies $DB \sim DB'$. Then $D \sim DA$ for all $A \in M$.

Proof. 1) follows since $M$ has multiplicative inverses. 2) and 3) are obvious from the definitions. QED

## II. 5 Hecke Operators

Before defining Hecke operators, we will prove a lemma which underlies the definition of Hecke operators, and which is essential to what follows.

<u>Lemma 1</u> (Standard Decomposition)

1) Let $\begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} \in \Gamma\frac{\underline{a}}{\underline{c}}(\mu)$. Then there exists $A \in \Gamma_{\underline{c}\,\underline{a}^{-1}}$ such that

$$\begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} = A \cdot C\frac{\underline{a}\,\kappa^{-1}}{\underline{c}\,\kappa^{-1}} \cdot \begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix}$$

with $\kappa, \sigma, \lambda \in O(Z)$, $\kappa\lambda = \mu$, $(\kappa)$ unique, $\sigma \bmod \lambda$ unique, and $[\sigma] = [\underline{a}^{-1}\underline{c}^{-1}\kappa^2\lambda]$.

2) If we choose particular $\kappa$ generating $(\kappa)$ for all $(\kappa) \mid (\mu)$, and if we choose particular $\sigma \bmod \lambda$ for $\lambda$ such that $\kappa\lambda = \mu$, then $A$ is unique.

<u>Proof.</u> Let $\kappa = (\pi, \psi)$, so $(\kappa)$ is unique and we can choose a particular generator $\kappa$. Let $\gamma = -\psi/\kappa$ and $\delta = \pi/\kappa$ so that $(\gamma, \delta) = 1$. By Prop. 1.2 we can find $\alpha, \beta \in O(Z)$ such that $\alpha\delta - \beta\gamma = 1$. Thus,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} = \begin{pmatrix} \kappa & \sigma' \\ 0 & \lambda \end{pmatrix}$$

where $\kappa\lambda = \mu$.

Choose $\nu \in O(Z)$ such that $\sigma' + \nu\lambda = \sigma$, where $\sigma$ is some chosen representative mod $\lambda$. Then

$$\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} = \begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix}$$

so

$$\begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} = \begin{pmatrix} \alpha + \nu\gamma & \beta + \nu\delta \\ \gamma & \delta \end{pmatrix}^{-1} \begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix} .$$

Using Prop. 2.2 we find that

$$\begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} = A \cdot C^{\underline{a}\,\kappa^{-1}}_{\underline{c}\,\kappa^{-1}} \cdot \begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix} \quad \text{for some } A \in \Gamma_{\underline{a}^{-1}\underline{c}} .$$

To show the uniqueness of $A$, suppose that

$$A_1 \cdot C^{\underline{a}\,\kappa_1^{-1}}_{\underline{c}\,\kappa_1^{-1}} \cdot \begin{pmatrix} \kappa_1 & \sigma_1 \\ 0 & \lambda_1 \end{pmatrix} = A_2 \cdot C^{\underline{a}\,\kappa_2^{-1}}_{\underline{c}\,\kappa_2^{-1}} \cdot \begin{pmatrix} \kappa_2 & \sigma_2 \\ 0 & \lambda_2 \end{pmatrix} .$$

If we let

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \left( C^{\underline{a}\,\kappa_2^{-1}}_{\underline{c}\,\kappa_2^{-1}} \right)^{-1} \cdot A_2^{-1} \cdot A_1 \cdot C^{\underline{a}\,\kappa_1^{-1}}_{\underline{c}\,\kappa_1^{-1}}$$

then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \kappa_1 & \sigma_1 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \kappa_2 & \sigma_2 \\ 0 & \lambda_2 \end{pmatrix} .$$

Thus $\gamma\kappa_1 = 0$ so $\gamma = 0$. Then $\alpha\delta = 1$ so $\alpha, \delta \in U$. Now $\alpha\kappa_1 = \kappa_2$ so $(\kappa_1) = (\kappa_2)$. By uniqueness of generator $\kappa_1 = \kappa_2$ which implies that $\alpha = 1$ and $\delta = 1$, as well as showing that

$\lambda_1 = \lambda_2$. One now sees that $\sigma_1 + \beta\lambda_1 = \sigma_2$ so $\sigma_1 \equiv \sigma_2 \mod \lambda_1$, but by uniqueness of choice of representative, $\sigma_1 = \sigma_2$. One concludes that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = I$ and thus $A_1 = A_2$, which verifies statement 2) of the lemma. <u>QED</u>

We are now almost ready to define Hecke operators. Let $\mu \in O(Z)$. Choose $\alpha_1 = 1, \alpha_2, \ldots, \alpha_h$ such that $\alpha_m \in O[\underline{a}_m]$ and $(\alpha_m, \mu) = 1$ for $m = 1, 2, \ldots, h$. This is possible by Prop. 1.1. For each $(\kappa)$ with $\kappa\lambda = \mu$, choose a particular generator $\kappa$. From each $O[\lambda]$, fix a complete set of representatives $\sigma_n \mod \lambda$, $n = 1, 2, \ldots, N(\lambda)$.

For convenience, define the vector operator $\left(\begin{pmatrix} \kappa & \sigma_n \\ 0 & \lambda \end{pmatrix}\right)$ to be $\left(B_{[\underline{a}_m]}\right)$ where

$$B_{[\underline{a}_m]} = \begin{pmatrix} \kappa & \alpha_m \sigma_n \\ 0 & \lambda \end{pmatrix} \in \Gamma^{\kappa}_{\kappa \underline{a}_m} .$$

<u>Proposition 5</u>

1) $\left(\begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix}\right) \cdot \left(\begin{pmatrix} \zeta & \omega \\ 0 & \xi \end{pmatrix}\right) \approx \left(\begin{pmatrix} \kappa\zeta & \kappa\omega + \xi\sigma \\ 0 & \lambda\xi \end{pmatrix}\right)$ .

2) $\left(\begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix}\right) \sim \left(\begin{pmatrix} \kappa & \sigma' \\ 0 & \lambda \end{pmatrix}\right)$ whenever $\sigma \equiv \sigma' \mod \lambda$ .

3) Given $B \in M(Z, \underline{I})$, $\lambda \in O(Z)$, we have

$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \cdot B = B \cdot \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) .$$

Proof. 1) One needs to observe that the consistency rule Prop. 2.1 is not violated; otherwise the proof is straightforward.

2) follows from 1), and 3) is elementary. QED

We can now define the unnormalized Hecke operators $t(\mu)$. In the next chapter, we will renormalize $t(\mu)$ but for this chapter it is convenient to ignore the normalizing constants.

Definition. The unnormalized Hecke operator $t(\mu)$ for $\mu \in O(Z)$ is given by

$$t(\mu) = \left(t_{[\underline{a}_m]}(\mu)\right) = \left( \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} C^{\kappa^{-1}}_{\kappa^{-1}\underline{a}_m} \sum_{n=1}^{N(\lambda)} \begin{pmatrix} \kappa & \alpha_p \sigma_n \\ 0 & \lambda \end{pmatrix} \right)$$

$$= \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} C(\kappa^{-1}) \cdot \sum_{n=1}^{N(\lambda)} \left( \begin{pmatrix} \kappa & \sigma_n \\ 0 & \lambda \end{pmatrix} \right)$$

where $\sigma_n \bmod \lambda$ is a fixed complete set of representatives and where $[\alpha_p] = [\underline{a}_m^{-1} \kappa^2]$.

Remark. Notice that $t(\mu) \in M(\mu, (1))$. We could also define a more general Hecke operator $t(\mu, \underline{b}) = C(\underline{b})\, t(\mu) \in M(\mu, \underline{b})$, but we will not need this generalization.

The following lemma shows that up to equivalence, $t(\mu)$ does not depend on the particular choices above.

<u>Lemma 2</u>

1) Replace $C^{\underline{b}}_{\underline{b}\,\underline{a}_m} \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}_m}$ by $C'^{\underline{b}}_{\underline{b}\,\underline{a}_m} \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}_m}$ for all $[\underline{b}]$ and $m = 1, 2, \ldots, h$. Call the new Hecke operator $t'(\mu)$. Then $C(\underline{b}) \sim C'(\underline{b})$ and $t(\mu) \sim t'(\mu)$.

2) Replace $\sigma_n \bmod \lambda$ by $\sigma_n' \bmod \lambda$, with $\sigma_n'$ in $O[\lambda]$ also a complete set of residues. Call the new Hecke operator $t'(\mu)$. Then

$$\sum_{n=1}^{N(\lambda)} \left(\begin{pmatrix} \kappa & \sigma_n \\ 0 & \lambda \end{pmatrix}\right) \sim \sum_{n=1}^{N(\lambda)} \left(\begin{pmatrix} \kappa & \sigma_n' \\ 0 & \lambda \end{pmatrix}\right)$$

and $t(\mu) \sim t'(\mu)$.

3) Replace $\alpha_m$ by $\beta_m$ with $\beta_m \in O[\underline{a}_m]$ and $(\beta_m, \mu) = 1$, $m = 1, 2, \ldots, h$. Call the new Hecke operator $t'(\mu)$. Then

$$\sum_{n=1}^{N(\lambda)} \begin{pmatrix} \kappa & \alpha_m \sigma_n \\ 0 & \lambda \end{pmatrix} \sim \sum_{n=1}^{N(\lambda)} \begin{pmatrix} \kappa & \beta_m \sigma_n \\ 0 & \lambda \end{pmatrix}$$

for $m = 1, 2, \ldots h$. Also, $t(\mu) \sim t'(\mu)$.

4) Replace $\kappa$ by $\kappa'$ where $(\kappa) = (\kappa')$ and $\kappa\lambda = \mu$ and $\kappa'\lambda' = \mu$. Let $\sigma_n \bmod \lambda$ and $\sigma_n' \bmod \lambda'$ be complete sets of representatives. Call the new Hecke operator (defined in terms of $\kappa'$, $\lambda'$, $\sigma_n'$) $t'(\mu)$. Then

$$\sum_{n=1}^{N(\lambda)} \left(\begin{pmatrix} \kappa & \sigma_n \\ 0 & \lambda \end{pmatrix}\right) \sim \sum_{n=1}^{N(\lambda)} \left(\begin{pmatrix} \kappa' & \sigma'_n \\ 0 & \lambda' \end{pmatrix}\right)$$

and $t(\mu) \sim t'(\mu)$.

<u>Proof</u>. 1) Prop. 2 shows that $C'^{\underline{b}}_{\underline{b}\,\underline{a}_m} = A_{[\underline{a}_m]} C^{\underline{b}}_{\underline{b}\,\underline{a}_m}$ for some $A_{[\underline{a}_m]} \in \Gamma_{\underline{a}_m}$. Thus $C(\underline{b}) \sim C'(\underline{b})$. Using Prop. 4.2, $t(\mu) \sim t'(\mu)$.

2) For each $n = 1, 2, \ldots, N(\lambda)$, there is a unique $p(n)$ such that $\sigma_n \equiv \sigma'_{p(n)} \bmod \lambda$. Using Prop. 5.1 and reordering the sum, we get the first equivalence. Then Prop. 4.3 with Prop. 3.5 shows that $t(\mu) \sim t'(\mu)$.

3) Fix any $m = 1, 2, \ldots, h$. By Lemma 2, there exists $A \in \Gamma_{\underline{a}_m}$ such that

$$A \cdot \begin{pmatrix} \kappa & \alpha_m \sigma_n \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \kappa' & \beta_m \sigma'_{p(n)} \\ 0 & \lambda' \end{pmatrix}$$

Uniqueness of the decomposition implies that

$$\kappa = \kappa', \ \lambda = \lambda', \ \text{and} \ \{\sigma_n\}_{n=1,2,\ldots,N(\lambda)} = \{\sigma_{p(n)}\}_{n=1,2,\ldots,N(\lambda)} .$$

We also find that $\alpha_m \sigma_n \equiv \beta_m \sigma_{p(n)} \bmod \lambda$, hence we can apply 2) above.

4) By hypothesis, $\kappa = \eta \kappa'$ for some $\eta \in U$, so $\lambda = \eta^{-1} \lambda'$. Using Prop. 5.1,

$$\left(\begin{pmatrix} \kappa & \sigma_n \\ 0 & \lambda \end{pmatrix}\right) = \left(\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}\right)\left(\begin{pmatrix} \kappa' & \eta^{-1}\sigma_n \\ 0 & \lambda' \end{pmatrix}\right) \sim \left(\begin{pmatrix} \kappa' & \eta^{-1}\sigma_n \\ 0 & \lambda' \end{pmatrix}\right)$$

Now $(\eta, \lambda) = 1$ for any $\eta \in U$, so Prop. 1.4 says that $\eta^{-1}\sigma_n$ is a complete set of representatives mod $\lambda'$. Applying 3) above, we are done. <u>QED</u>

<u>Theorem 1</u> (Invariance of the Hecke operator).

$$B \cdot t(\mu) \sim t(\mu) \cdot B \quad \text{for any } B \in M(1, \underline{I}) .$$

<u>Proof.</u> Recall that $t(\mu) = \left(t_{[\underline{a}_m]}(\mu)\right)$ where

$$t_{[\underline{a}_m]}(\mu) = \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} C^{\kappa^{-1}}_{\kappa^{-1}\underline{a}_m} \sum_{n=1}^{N(\lambda)} \begin{pmatrix} \kappa & \alpha_p \sigma_n \\ 0 & \lambda \end{pmatrix}$$

where $[\alpha_p] = [\kappa^2 \underline{a}_m^{-1}]$. Let $D \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}_m \mu^{-1}}$. We want to show that multiplying $t_{[\underline{a}_m]}(\mu)$ on the right by $D$ will effectively permute the matrices $\begin{pmatrix} \kappa & \alpha_p \sigma_n \\ 0 & \lambda \end{pmatrix}$.

$$\text{Set } \begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} = C^{\kappa^{-1}}_{\kappa^{-1}\underline{a}_m} \begin{pmatrix} \kappa & \alpha_p \sigma_n \\ 0 & \lambda \end{pmatrix} \cdot D \quad \text{so} \quad \begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}_m}(\mu).$$

By Lemma 1 there exists a unique $A \in \Gamma_{\underline{a}_m}$ such that

$$\begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} = A \cdot C^{\underline{b}\,\kappa'^{-1}}_{\underline{b}\,\underline{a}_m \kappa'^{-1}} \cdot \begin{pmatrix} \kappa' & \alpha_r \sigma_s' \\ 0 & \lambda' \end{pmatrix}$$

where we fix representatives $\sigma_s'$ mod $\lambda'$ and where $[\alpha_r] = [\underline{b}^{-1} \underline{a}_m^{-1} \kappa'^2]$. I claim that each term $\begin{pmatrix} \kappa & \alpha_p \sigma_n \\ 0 & \lambda \end{pmatrix}$ of $t_{[\underline{a}_m]}$ gives rise to a distinct term $\begin{pmatrix} \kappa' & \alpha_r \sigma_s' \\ 0 & \lambda' \end{pmatrix}$. If this is true, then Prop. 2.4 shows that

$$t_{[\underline{a}_m]}(\mu) \cdot D \sim C^{\underline{b}}_{\underline{b}\,\underline{a}_m} \sum_{\substack{(\kappa') \\ \kappa'\lambda' = \mu}} C^{\kappa'^{-1}}_{\kappa'^{-1} \underline{a}_m} \cdot \sum_{s=1}^{N(\lambda')} \begin{pmatrix} \kappa' & \alpha_r \sigma_s' \\ 0 & \lambda' \end{pmatrix}$$

and using Prop. 2.2 we may conclude that

$$t_{[\underline{a}_m]}(\mu) \cdot D \sim D' \cdot t_{[\underline{a}_m]}(\mu) \quad \text{for any } D' \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}_m} .$$

This means that if $B = \left(B_{[\underline{a}_m]}\right)$ and if we let $D = B_{[\underline{a}_m \mu^{-1}]}$, $D' = B_{[\underline{a}_m]}$, then

$$t(\mu) \cdot B \sim B \cdot t(\mu) .$$

We need to show that the matrices $\begin{pmatrix} \kappa' & \alpha_r \sigma_s' \\ 0 & \lambda' \end{pmatrix}$ are distinct. Suppose we have

$$C^{\kappa_1^{-1}}_{\kappa_1^{-1}\underline{a}_m} \cdot \begin{pmatrix} \kappa_1 & \alpha_p \sigma_1 \\ 0 & \lambda_1 \end{pmatrix} \cdot D = A_1 \cdot C^{\underline{b}\,\kappa^{-1}}_{\underline{b}\,\underline{a}_m \kappa^{-1}} \cdot \begin{pmatrix} \kappa & \alpha_r \sigma_s \\ 0 & \lambda \end{pmatrix}$$

and also

$$C^{\kappa_2^{-1}}_{\kappa_2^{-1}\underline{a}_m} \cdot \begin{pmatrix} \kappa_2 & \alpha_q \sigma_2 \\ 0 & \lambda_2 \end{pmatrix} \cdot D = A_2 \cdot C^{\underline{b}\,\kappa^{-1}}_{\underline{b}\,\underline{a}_m\,\kappa^{-1}} \cdot \begin{pmatrix} \kappa & \alpha_r \sigma_s \\ 0 & \lambda \end{pmatrix} .$$

We wish to show that $\kappa_1 = \kappa_2$, $\lambda_1 = \lambda_2$, and $\alpha_p \sigma_1 = \alpha_q \sigma_2$. Simple matrix manipulation shows that

$$\begin{pmatrix} \kappa_1 & \alpha_p \sigma_1 \\ 0 & \lambda_1 \end{pmatrix} = \left( C^{\kappa_1^{-1}}_{\kappa_1^{-1}\underline{a}_m} \right)^{-1} \cdot A_1 \cdot A_2^{-1} \cdot C^{\kappa_2^{-1}}_{\kappa_2^{-1}\underline{a}_m} \begin{pmatrix} \kappa_2 & \alpha_q \sigma_2 \\ 0 & \lambda_2 \end{pmatrix} .$$

Using Prop. 2.4, 2.5, and 2.6, there exists $A_3 \in \Gamma_{\underline{a}_m \kappa_1^{-1}}$ such that

$$\begin{pmatrix} \kappa_1 & \alpha_p \sigma_1 \\ 0 & \lambda_1 \end{pmatrix} = A_3 \cdot C^{\kappa_1 \kappa_2^{-1}}_{\kappa_1^{-1}\kappa_2^{-1}\underline{a}_m} \cdot \begin{pmatrix} \kappa_2 & \alpha_q \sigma_2 \\ 0 & \lambda_2 \end{pmatrix} .$$

Both sides are in the standard form of Lemma 1, so by the uniqueness statements of Lemma 1,

$$\begin{pmatrix} \kappa_1 & \alpha_p \sigma_1 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \kappa_2 & \alpha_q \sigma_2 \\ 0 & \lambda_2 \end{pmatrix} .$$

<u>QED</u>

We are now ready for the key theorem of this chapter.

Theorem 2 (Multiplicative relations of Hecke operators).

$$t(\mu)\,t(\nu) = \sum_{\substack{(\theta) \\ \theta \mid (\mu,\nu)}} N(\theta)\, C(\theta^{-1}) \left(\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}\right) t\left(\frac{\mu\nu}{\theta^2}\right) .$$

Proof. First we will show that this relation holds for all $(\mu, \nu) = 1$, then for prime powers, and finally for arbitrary $\mu$ and $\nu$ in $O(Z)$.

Suppose $(\mu, \nu) = 1$. We wish to show that $t(\mu)\,t(\nu) = t(\mu\nu)$.

Let

$$t(\mu) = \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} C(\kappa^{-1}) \sum_{n=1}^{N(\lambda)} \left(\begin{pmatrix} \kappa & \sigma_n \\ 0 & \lambda \end{pmatrix}\right)$$

and

$$t(\nu) = \sum_{\substack{(\zeta) \\ \zeta\xi = \nu}} C(\zeta^{-1}) \sum_{r=1}^{N(\xi)} \left(\begin{pmatrix} \zeta & \omega_r \\ 0 & \xi \end{pmatrix}\right) .$$

Then

$$t(\mu)\,t(\nu) = t(\mu) \sum_{(\zeta)} C(\zeta^{-1}) \sum_{r=1}^{N(\xi)} \left(\begin{pmatrix} \zeta & \omega_r \\ 0 & \xi \end{pmatrix}\right)$$

$$\sim \sum C(\zeta^{-1})\, t(\mu) \sum \left(\begin{pmatrix} \zeta & \omega_r \\ 0 & \xi \end{pmatrix}\right)$$

using Theorem 1. Note that I have also used Prop. 4.2. I will be using Prop. 4.2 and 4.3 continually throughout this proof, and will generally not mention their use.

$$t(\mu)\, t(\nu) \sim \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} \sum_{\substack{(\zeta) \\ \zeta\xi = \nu}} C(\kappa^{-1}) \cdot C(\zeta^{-1}) \cdot \sum_{n=1}^{N(\lambda)} \sum_{r=1}^{N(\xi)} \left(\!\!\begin{pmatrix} \kappa & \sigma_n \\ 0 & \lambda \end{pmatrix}\!\!\right)\left(\!\!\begin{pmatrix} \zeta & \omega_r \\ 0 & \xi \end{pmatrix}\!\!\right).$$

Using Prop. 3.6 and Prop. 5.1 we find that

$$t(\mu)\, t(\nu) \sim \sum_{(\kappa)} \sum_{(\zeta)} C((\kappa\zeta)^{-1}) \sum_{n} \sum_{r} \left(\!\!\begin{pmatrix} \kappa\zeta & \kappa\omega_r + \sigma_n \xi \\ 0 & \lambda\xi \end{pmatrix}\!\!\right)$$

I claim that $\kappa\omega_r + \sigma_n \xi$ is a complete set of representatives mod $(\lambda\xi)$ if $\sigma_n$ mod $\lambda$ and $\omega_r$ mod $\xi$ are complete. Further, note that $[\kappa\omega_r + \sigma_n \xi] = [\lambda\xi]$. Assuming that my claim is true, Lemma 2 shows that the right side is $t(\mu\nu)$ and we are done.

Suppose $\sigma_n$ mod $\lambda$ and $\omega_r$ mod $\xi$ are complete sets of representatives, with $[\sigma_n] = [\lambda]$ and $[\omega_r] = [\xi]$. There are $N(\lambda\xi)$ representatives mod $\lambda\xi$ and I want to show that if $\kappa\omega_r + \sigma_n\xi \equiv \kappa\omega_s + \sigma_p\xi$ mod $\lambda\xi$ then $\omega_r = \omega_s$ and $\sigma_n = \sigma_p$.

Clearly $\kappa\omega_r \equiv \kappa\omega_s$ mod $\xi$. By hypothesis, $(\mu, \nu) = 1$ so $(\kappa, \xi) = 1$. Using Prop. 1.5, $\omega_r \equiv \omega_s$ mod $\xi$ hence $\omega_r = \omega_s$. Then $\sigma_n\xi \equiv \sigma_p\xi$ mod $\lambda\xi$ so $\sigma_n \equiv \sigma_p$ mod $\lambda$ which says that $\sigma_n = \sigma_p$.

Thus we have finished showing that $t(\mu)t(\nu) \sim t(\mu\nu)$ whenever $(\mu, \nu) = 1$. Now we wish tc show that for $\pi$ prime, $t(\pi)\,t(\pi^r)$ $\sim t(\pi^{r+1}) + N(\pi)\, C(\pi^{-1}) \left(\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}\right) t(\pi^{r-1})$ for any $r$ a positive integer.

Let

$$t(\pi) = C(\pi^{-1}) \left(\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{n=1}^{N(\pi)} \left(\begin{pmatrix} 1 & \sigma_n \\ 0 & \pi \end{pmatrix}\right)$$

and

$$t(\pi^r) = \sum_{s=0}^{r} C(\pi^{-s}) \sum_{u=1}^{N(\pi^{r-s})} \left(\begin{pmatrix} \pi^s & \omega_u \\ 0 & \pi^{r-s} \end{pmatrix}\right)$$

By Theorem 1,

$$t(\pi)\, t(\pi^r) \sim \sum_{s=0}^{r} C(\pi^{-s})\, t(\pi) \sum_{u=1}^{N(\pi^{r-s})} \left(\begin{pmatrix} \pi^s & \omega_u \\ 0 & \pi^{r-s} \end{pmatrix}\right)$$

$$= \sum_{s=0}^{r} C(\pi^{-s})\, C(\pi^{-1}) \sum_{u=1}^{N(\pi^{r-s})} \left(\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} \pi^s & \omega_u \\ 0 & \pi^{r-s} \end{pmatrix}\right)$$

$$+ \sum_{s=0}^{r} C(\pi^{-s}) \sum_{n=1}^{N(\pi)} \sum_{u=1}^{N(\pi^{r-s})} \left(\begin{pmatrix} 1 & \sigma_n \\ 0 & \pi \end{pmatrix}\right)\left(\begin{pmatrix} \pi^s & \omega_u \\ 0 & \pi^{r-s} \end{pmatrix}\right)$$

$$\sim \sum_{s=0}^{r-1} C(\pi^{-s})\, C(\pi^{-1}) \sum_{u=1}^{N(\pi^{r-s})} \left(\!\!\begin{pmatrix} \pi^{s+1} & \pi\omega_u \\ 0 & \pi^{r-s} \end{pmatrix}\!\!\right)$$

$$+\ C(\pi^{r+1}) \left(\!\!\begin{pmatrix} \pi^{r+1} & 0 \\ 0 & 1 \end{pmatrix}\!\!\right)$$

$$+ \sum_{s=0}^{r} C(\pi^{-s}) \sum_{n=1}^{N(\pi)} \sum_{u=1}^{N(\pi^{r-s})} \left(\!\!\begin{pmatrix} \pi^{s} & \omega_u + \sigma_n \pi^{r-s} \\ 0 & \pi^{r-s+1} \end{pmatrix}\!\!\right)$$

Just as above, we find that there are $N(\pi^{r-s+1})$ distinct $\omega_u + \sigma_n \pi^{r-s} \bmod \pi^{r-s+1}$. Thus Lemma 2.2 allows us to combine the last two expressions, and we conclude that

$$t(\pi)\, t(\pi^r) \sim \sum_{s=0}^{r-1} C(\pi^{-s})\, C(\pi^{-1}) \sum_{u=1}^{N(\pi^{r-s})} \left(\!\!\begin{pmatrix} \pi^{s+1} & \pi\omega_u \\ 0 & \pi^{r-s} \end{pmatrix}\!\!\right) + t(\pi^{r+1}).$$

Decompose $\omega_u = \tau_v + \sigma_n \pi^{r-s-1}$ where $\tau_v \bmod \pi^{r-s-1}$ and $\sigma_n \bmod \pi$ are unique. Since $N(\pi^{r-s}) = N(\pi^{r-s-1})\, N(\pi)$, we see that $\tau_v \bmod \pi^{r-s-1}$ and $\sigma_n \bmod \pi$ are complete sets of representatives. Thus

$$t(\pi)\, t(\pi^r) \sim t(\pi^{r+1}) + \sum_{s=1}^{r-1} C(\pi^{-s})\, C(\pi^{-1}) \left(\!\!\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}\!\!\right) .$$

$$\cdot \sum_{n=1}^{N(\pi)} \sum_{v=1}^{N(\pi^{r-s-1})} \left(\begin{pmatrix} \pi^s & \tau_v + \sigma_n \pi^{r-s-1} \\ 0 & \pi^{r-s-1} \end{pmatrix}\right)$$

Using Prop. 5.3 and 3.6, we can interchange terms and apply Lemma 2.2 to get

$$t(\pi)\, t(\pi^r) \sim t(\pi^{r+1}) +$$

$$+ \sum_{n=1}^{N(\pi)} C(\pi^{-1}) \left(\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}\right) \sum_{s=0}^{r-1} C(\pi^{-s}) \sum_{v=1}^{N(\pi^{r-s-1})} \left(\begin{pmatrix} \pi^s & \tau_v \\ 0 & \pi^{r-s-1} \end{pmatrix}\right)$$

$$\sim t(\pi^{r+1}) + N(\pi)\, C(\pi^{-1}) \left(\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}\right) t(\pi^{r-1}) \quad .$$

Now we will use induction to show that

$$t(\pi^q)\, t(\pi^r) \sim \sum_{s=0}^{\min(q,r)} N(\pi^s)\, C(\pi^{-s}) \left(\begin{pmatrix} \pi^s & 0 \\ 0 & \pi^s \end{pmatrix}\right) t(\pi^{q+r-2s}) \quad .$$

Assume this is true for $q - 1$ and all $r$, with $q \geq 2$. We have just shown that it is true for $q = 1$.

$$t(\pi^q)\, t(\pi^r) \sim \left[ t(\pi)\, t(\pi^{q-1}) - N(\pi)\, C(\pi^{-1}) \left(\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}\right) \cdot t(\pi^{q-2}) \right] \cdot t(\pi^r)$$

by the above. Applying the inductive hypothesis,

$$t(\pi^q)\, t(\pi^r) \sim t(\pi) \cdot \sum_{s=0}^{\min(q-1,r)}$$

$$\cdot\ N(\pi^s)\, C(\pi^{-s}) \left(\begin{pmatrix} \pi^s & 0 \\ 0 & \pi^s \end{pmatrix}\right) t(\pi^{q+r-1-2s})$$

$$-\ N(\pi)\, C(\pi^{-1}) \left(\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}\right) \sum_{s=0}^{\min(q-2,r)} N(\pi^s)\, C(\pi^{-s}) \left(\begin{pmatrix} \pi^s & 0 \\ 0 & \pi^s \end{pmatrix}\right) \cdot$$

$$\cdot\ t(\pi^{q+r-2-2s})$$

$$\sim \sum_{s=0}^{\min(q-1,r)} N(\pi^s)\, C(\pi^{-s}) \left(\begin{pmatrix} \pi^s & 0 \\ 0 & \pi^s \end{pmatrix}\right) \Bigg[ t(\pi^{q+r-2s})$$

$$+\ N(\pi)\, C(\pi^{-1}) \left(\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}\right) t(\pi^{q+r-2-2s}) \Bigg]$$

$$-\sum_{s=0}^{\min(q-2,r)} N(\pi^{s+1})\, C(\pi^{-s-1}) \left(\begin{pmatrix} \pi^{s+1} & 0 \\ 0 & \pi^{s+1} \end{pmatrix}\right) \cdot t(\pi^{q+r-2-2s})$$

where I have used Prop. 5.3 and Prop. 4 and Prop. 3.6 several times.

Finally, define $f(q,r) = 1$ if $r \geq q-1$ and $f(q,r) = 0$ if $r < q-1$. Then the above becomes

$$t(\pi^q)\,t(\pi^r) \sim \sum_{s=0}^{\min(q-1,r)} N(\pi^s)\, C(\pi^{-s}) \left(\begin{pmatrix} \pi^s & 0 \\ 0 & \pi^s \end{pmatrix}\right) t(\pi^{q+r-2s})$$

$$+ f(q,r)\; N(\pi^q)\, C(\pi^{-q}) \left(\begin{pmatrix} \pi^q & 0 \\ 0 & \pi^q \end{pmatrix}\right) t(\pi^{r-q}) \quad .$$

This implies that

$$t(\pi^q)\,t(\pi^r) \sim \sum_{s=0}^{\min(q,r)} N(\pi^s)\, C(\pi^{-s}) \left(\begin{pmatrix} \pi^s & 0 \\ 0 & \pi^s \end{pmatrix}\right) t(\pi^{q+r-2s})$$

which is the desired result.

We still need to consider the general case of $\mu$ and $\nu$ with $(\mu, \nu) > 1$. First we notice that the expressions we have already shown imply commutativity. By Prop. 1.6 we can factor $\mu$ and $\nu$ into prime factors, where we can fix a particular generator $\pi$ for any prime ideal $(\pi)$ dividing $(\mu)$ and $(\nu)$. We have just finished showing that the result is true for each prime factor, and then we can use our statement about relatively prime $t(\mu)$ and $t(\nu)$ to conclude the proof. QED

II. 6 Example: Hecke Operators for $K = \mathbb{Q}(\sqrt{-23})$

First we will define a set of ideal numbers. It is well known that $\mathbb{Q}(\sqrt{-23})$ has class number 3 and that either prime ideal dividing (2) generates the ideal class group. We can easily calculate a set of ideal numbers Z corresponding to $\mathbb{Q}(\sqrt{-23})$.

Let $\underline{p}_2 = \left(\frac{1 - \sqrt{-23}}{2}, 2\right)$ and $\underline{p}_2 = \left(\frac{1 + \sqrt{-23}}{2}, 2\right)$. We will choose $\underline{p}_2$ as the generator of the ideal class group. Now $\underline{p}_2^3 = \left(\frac{3 + \sqrt{-23}}{2}\right)$ so we choose some branch of the cube root and define an ideal number $\pi_2 = \sqrt[3]{\frac{3 + \sqrt{-23}}{2}}$ which corresponds to the ideal $\underline{p}_2$. Because the choice of generator of $\left(\frac{3 + \sqrt{-23}}{2}\right)$ has an ambiguity of sign, and because the choice of cube root is arbitrary, $\pi_2$ contains an arbitrary sixth root of unity. We fix this choice, however, so the rest of the ideal numbers are determined.

For any ideal $\underline{a}$, if $\underline{a} = (a)$ is principal, then the corresponding ideal numbers are $\pm a$. If $\underline{p}_2\underline{a} = (a)$ is principal, then the ideal numbers corresponding to $\underline{a}$ are $\pm a/\pi_2$. If $\underline{p}_2^2\underline{a} = (a)$ is principal, then the ideal numbers corresponding to $\underline{a}$ are $\pm a/\pi_2^2 = \pm a \cdot \pi_2 \bigg/ \left(\frac{3 + \sqrt{-23}}{2}\right)$.

For convenience, let $b = \sqrt{-23}$. One can verify the following facts.

$$\left(\frac{1+b}{2}\right) = \overline{\underline{p}}_2 \underline{p}_3 \qquad \left(\frac{1-b}{2}\right) = \underline{p}_2 \overline{\underline{p}}_3$$

$$\left(\frac{9+b}{2}\right) = \overline{\underline{p}}_2 \underline{p}_{13} \qquad \left(\frac{9-b}{2}\right) = \underline{p}_2 \overline{\underline{p}}_{13}$$

$$\left(\frac{5 - 3b}{2}\right) = \overline{\underline{p}}_2 \underline{p}_{29} \qquad \left(\frac{5 + 3b}{2}\right) = \underline{p}_2 \overline{\underline{p}}_{29}$$

$$\left(\frac{15 - b}{2}\right) = \overline{\underline{p}}_2 \underline{p}_{31} \qquad \left(\frac{15 + b}{2}\right) = \underline{p}_2 \overline{\underline{p}}_{31}$$

$$\left(\frac{11 + 3b}{2}\right) = \overline{\underline{p}}_2 \underline{p}_{41} \qquad \left(\frac{11 - 3b}{2}\right) = \underline{p}_2 \overline{\underline{p}}_{41}$$

where $\underline{p}_q$ are defined consistent with the following table:

| $q$ | $\sqrt{-23} \bmod \underline{p}_q$ | $\sqrt{-23} \bmod \overline{\underline{p}}_q$ |
|---|---|---|
| 3 | -1 | 1 |
| 13 | 4 | 9 |
| 29 | 21 | 8 |
| 31 | 15 | 16 |
| 41 | 10 | 31 |

Now we can define corresponding ideal numbers. Define the following:

$$\pi_2 = \sqrt[3]{\frac{3 + b}{2}} \qquad \overline{\pi}_2 = \frac{2}{\pi_2}$$

$$\pi_3 = \frac{1 + b}{4} \cdot \pi_2 \qquad \overline{\pi}_3 = \frac{1 - b}{2\pi_2}$$

$$\pi_{13} = \frac{9 + b}{4} \cdot \pi_2 \qquad \overline{\pi}_{13} = \frac{9 - b}{2\pi_2}$$

$$\pi_{29} = \frac{5-3b}{4} \cdot \pi_2 \qquad \bar{\pi}_{29} = \frac{5+3b}{2\pi_2}$$

$$\pi_{31} = \frac{15-b}{4} \cdot \pi_2 \qquad \bar{\pi}_{31} = \frac{15+b}{2\pi_2}$$

$$\pi_{41} = \frac{11+3b}{4} \cdot \pi_2 \qquad \bar{\pi}_{41} = \frac{11-3b}{2\pi_2}$$

Here $\pi_q$ corresponds to $\underline{p}_q$ and $\bar{\pi}_q$ corresponds to $\underline{\bar{p}}_q$ .

We can now calculate some multiplicative relations we will use later.

$$\pi_q \cdot \bar{\pi}_q = q \quad \text{for} \quad q = 2,\ 3,\ 13,\ 29,\ 31,\ 41.$$

$$\pi_2^3 = \frac{3+b}{2} \qquad \bar{\pi}_2^3 = \frac{3-b}{2}$$

$$\pi_2\pi_3^2 = -\frac{7+b}{2} \qquad \bar{\pi}_2\pi_3 = \frac{1+b}{2}$$

$$\pi_3^3 = 2-b \qquad \bar{\pi}_3^3 = 2+b$$

$$\bar{\pi}_2^2\pi_3^2 = \frac{-11+b}{2} \qquad \pi_{13}^3 = 37+6b$$

$$\bar{\pi}_2\pi_{13} = \frac{9+b}{2} \qquad \pi_3\pi_{13}^2 = -22-b$$

$$\pi_2 \cdot \pi_3\pi_{13} = \frac{-17+d}{2}$$

For convenience, let $O_0 = O[(1)]$, $O_1 = O[\pi_2]$, and $O_2 = O[\pi_2^2]$.

As a typical example, we will find the Hecke operator $t(\pi_{13})$. To do this, we need to fix representatives mod $\pi_{13}$ and also fix our "standard matrices" $C(\underline{a})$. For convenience we will denote $\Gamma_{\pi_2^b}^{\pi_2^a}$ by $\Gamma_b^a$ and $C_{\pi_2^b}^{\pi_2^a}$ by $C_b^a$, where $a$ and $b$ may be considered mod 3, since the class number is 3. We will also denote $t_{[\pi_2^a]}(\pi_{13})$ by $t_a(\pi_{13})$ so that

$$t(\pi_{13}) = \begin{pmatrix} t_0(\pi_{13}) \\ t_1(\pi_{13}) \\ t_2(\pi_{13}) \end{pmatrix} .$$

As a choice of representatives mod $\pi_{13}$ in $O_0$, take $0, 1, 2, \ldots, 12$. In $O_1$, take representatives $0, \pi_2, 2\pi_2, \ldots, 12\pi_2$, and in $O_2$ take $0, \pi_2^2, 2\pi_2^2, \ldots, 12\pi_2^2$.

For the $C_b^a$, we can consider "variations" of the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \pi_3 & \bar{\pi}_2 \\ \pi_2 & \bar{\pi}_3 \end{pmatrix}$. In particular, let $C_m^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $m = 0, 1, 2$, $C_0^m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ for $m = 1, 2$,

$$C_1^1 = \begin{pmatrix} \pi_3 & \bar{\pi}_2 \\ \pi_2 & \bar{\pi}_3 \end{pmatrix}, \quad C_2^1 = \begin{pmatrix} \pi_3 & \pi_2 \\ \bar{\pi}_2 & \bar{\pi}_3 \end{pmatrix},$$

$$C_1^2 = \begin{pmatrix} \bar{\pi}_3 & \bar{\pi}_2 \\ \pi_2 & \pi_3 \end{pmatrix}, \quad \text{and} \quad C_2^2 = \begin{pmatrix} \bar{\pi}_3 & \pi_2 \\ \bar{\pi}_2 & \pi_3 \end{pmatrix}.$$

Now we can define $t(\pi_{13})$ by letting

$$t_0(\pi_{13}) = \begin{pmatrix} \bar{\bar{\pi}}_3\pi_{13} & \pi_2 \\ \bar{\pi}_2\pi_{13} & \pi_3 \end{pmatrix} + \sum_{q=0}^{12} \begin{pmatrix} 1 & q\pi_2 \\ 0 & \pi_{13} \end{pmatrix}$$

and $$t_1(\pi_{13}) = \begin{pmatrix} 0 & -1 \\ \pi_{13} & 0 \end{pmatrix} + \sum_{s=0}^{12} \begin{pmatrix} 1 & s \\ 0 & \pi_{13} \end{pmatrix}$$

and $$t_2(\pi_{13}) = \begin{pmatrix} \bar{\bar{\pi}}_3\pi_{13} & \bar{\bar{\pi}}_2 \\ \pi_2\pi_{13} & \pi_3 \end{pmatrix} + \sum_{r=0}^{12} \begin{pmatrix} 1 & r\pi_2^2 \\ 0 & \pi_{13} \end{pmatrix}.$$

Consider the principal class component of $t(\pi_{13})^3$. This component is simply $t_0(\pi_{13}) \cdot t_2(\pi_{13})\, t_1(\pi_{13})$. Multiplying out the matrices directly yields the following expression:

$$\begin{pmatrix} \bar{\bar{\pi}}_2\bar{\bar{\pi}}_3\pi_{13}^2 + \pi_2\pi_3\pi_{13} & -(\bar{\bar{\pi}}_3\pi_{13}^2 + \pi_2^2\pi_{13}) \\ \bar{\bar{\pi}}_2^2\pi_{13}^2 + \pi_3^2\pi_{13} & -\bar{\bar{\pi}}_2\bar{\bar{\pi}}_3\pi_{13}^2 \end{pmatrix}$$

$$+ \sum_{q=0}^{12} \begin{pmatrix} \bar{\bar{\pi}}_2\pi_{13} + q\pi_2\pi_3\pi_{13} & -(\bar{\bar{\pi}}_3\pi_{13} + q\pi_2^2\pi_{13}) \\ \pi_3\pi_{13}^2 & -\pi_2\pi_{13}^2 \end{pmatrix}$$

$$+\sum_{r=0}^{12}\begin{pmatrix}\pi_2\pi_{13}^2+r\pi_2^2\bar{\pi}_3\pi_{13}^2 & -\bar{\pi}_3\pi_{13}\\ \pi_3\pi_{13}^2+2r\pi_2\pi_{13}^2 & -\bar{\pi}_2\pi_{13}\end{pmatrix}$$

$$+\sum_{s=0}^{12}\begin{pmatrix}\bar{\pi}_3^2\pi_{13}^2+\pi_2^2\pi_{13} & \bar{\pi}_2\bar{\pi}_3\pi_{13}^2+\pi_2\pi_3\pi_{13}+s(\bar{\pi}_3^2\pi_{13}^2+\pi_2^2\pi_{13})\\ \bar{\pi}_2\bar{\pi}_3\pi_{13}^2+\pi_2\pi_3\pi_{13} & \bar{\pi}_2^2\pi_{13}^2+\pi_3^3+s(\bar{\pi}_2\bar{\pi}_3\pi_{13}^2+\pi_2\pi_3\pi_{13})\end{pmatrix}$$

$$+\sum_{q=0}^{12}\sum_{r=0}^{12}\begin{pmatrix}q\pi_2\pi_{13}^2+r\pi_2^2\pi_{13} & -1\\ \pi_{13}^3 & 0\end{pmatrix}$$

$$+\sum_{q=0}^{12}\sum_{s=0}^{12}\begin{pmatrix}\bar{\pi}_3\pi_{13}+q\pi_2^2\pi_{13} & \bar{\pi}_2\pi_{13}+q\pi_2\pi_3\pi_{13}+s(\bar{\pi}_3\pi_{13}+q\pi_2^2\pi_{13})\\ \pi_2\pi_{13}^2 & \pi_3\pi_{13}^2+s\pi_2\pi_{13}^2\end{pmatrix}$$

$$+\sum_{r=0}^{12}\sum_{s=0}^{12}\begin{pmatrix}\bar{\pi}_3\pi_{13} & \pi_2\pi_{13}^2+r\pi_2^2\bar{\pi}_3\pi_{13}^2+s\bar{\pi}_3\pi_{13}\\ \bar{\pi}_2\pi_{13} & \pi_3\pi_{13}^2+2r\pi_2\pi_{13}^2+s\bar{\pi}_2\pi_{13}\end{pmatrix}$$

$$+\sum_{q=0}^{12}\sum_{r=0}^{12}\sum_{s=0}^{12}\begin{pmatrix}1 & q\pi_2\pi_{13}^2+r\pi_2^2\pi_{13}+s\\ 0 & \pi_{13}^3\end{pmatrix}$$

Notice that every entry is a principal element. In other words, ideal numbers are not needed to define the principal component of $t(\pi_{13})^3$.

## III. 1 The Quaternionic Upper Half Space

The classical theory of modular forms considers functions defined on the upper half plane which is acted on by $SL(2,\mathbb{R})$. As one might expect, mathematicians quickly generalized the basic notions. One of the earliest generalizations was the Poincaré 3-space, also known as the quaternionic upper half space.

Let $i = \sqrt{-1}, j$, and $i \cdot j = k$ be the unit basis elements for the quaternion algebra $Q$. One defines conjugation as follows: $q = a + bi + cj + dk \in Q$ has conjugate $\overline{q} = a - bi - cj - dk \in Q$ for any $a, b, c, d \in \mathbb{R}$. One defines the norm of $q$ to be $|q| = (a^2 + b^2 + c^2 + d^2)^{1/2}$. The following proposition summarizes trivial but useful properties of $Q$.

Proposition 6

1) $\overline{(q_1 q_2)} = \overline{q}_2 \overline{q}_1$ for any $q_1, q_2 \in Q$ .

2) $q^{-1} = \overline{q}/|q|^2$ for any $q \in Q - \{0\}$ .

3) $\alpha k = k\overline{\alpha}$ for any $\alpha \in \mathbb{C}$ .

Proof. This is left to the reader. QED

Define the quaternionic upper half space $H$ by $H = \{z \in Q \mid z = x + yk,\ x \in \mathbb{C},\ y > 0\}$. Thus $H \cong \mathbb{C} \times \mathbb{R}^+$. We define an action of $SL(2,\mathbb{C})$ on $H$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) = (\alpha z + \beta)(\gamma z + \delta)^{-1}$$

for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{C})$ and $z \in H$.

Proposition 7

1) The action is well-defined, that is, if $A \in SL(2, \mathbb{C})$ and $z \in H$, then $A(z) \in H$.

2) For any $A, B \in SL(2, \mathbb{C})$, $A(B(z)) = (AB)(z)$.

3) The maximal subgroup of $SL(2, \mathbb{C})$ fixing $z = k$ is $SU(2)$.

4) $SL(2, \mathbb{C})$ acts transitively on $H$.

Proof. All of these have slick proofs (see Siegel [ 9 ]) but also follow from straightforward calculation using Prop. 6. I will not need 3) and 4) so will not prove them, and 1) and 2) are simple.

QED

## III.2 Quaternionic Representations

Quaternions do not commute, which discourages consideration of quaternion-valued functions. Using representations, however, one can obtain complex-valued functions with a quaternion argument.

Let $Mat(n, \mathbb{C})$ be the algebra of all $n \times n$ complex matrices. The basic representation $\rho$ of the quaternions $Q$ is given by:

$$\rho : Q \to Mat(2, \mathbb{C})$$

$$\rho(x + yk) = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \quad \text{for } x, y \in \mathbb{C} .$$

We construct other representations of $Q$ by considering symmetric powers. Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a matrix, and $u, v$ variables. Then define $u', v'$ by $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. Let the n-fold symmetric product of $A$ be the $(n+1) \times (n+1)$ matrix $A^{(n)}$ given by

$$\begin{pmatrix} u'^{n} \\ u'^{n-1} v' \\ u'^{n-2} v'^{2} \\ \vdots \\ v'^{n} \end{pmatrix} = A^{(n)} \begin{pmatrix} u^{n} \\ u^{n-1} v \\ u^{n-2} v^{2} \\ \vdots \\ v^{n} \end{pmatrix}$$

Now we define the n-fold symmetric quaternion representation

$$\rho^{n} : Q \to \mathrm{Mat}(n+1, \mathbb{C}) \qquad \text{by}$$

$$\rho^{n}(x + yk) = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}^{(n)} .$$

Notice that $\rho^{n}$ is actually an abusive notation; for instance, $\rho^{2}$ has nothing to do with $\rho \circ \rho$. To further abuse notation, let $\rho^{-n}(x + yk) = (\rho^{n}(x + yk))^{-1}$.

Proposition 8

1) $\rho^n$ is a representation, that is, for any $q_1, q_2 \in Q$,

$\rho^n(q_1 q_2) = \rho^n(q_1)\, \rho^n(q_2)$.

2) $\rho^{-n}(q) = \rho^n(q^{-1})$ .

3) $\rho^n(rq) = r^n \rho^n(q)$ for any $r \in \mathbb{R}$ .

4) $\rho^{-n}(q_1 q_2) = \rho^{-n}(q_2)\, \rho^{-n}(q_1)$ .

Proof. 1) $\rho^n$ will be a representation if both $\rho$ and the n-fold symmetric product are representations. Simple calculation shows that $\rho$ is a representation. If we let

$$\begin{pmatrix} u'' \\ v'' \end{pmatrix} = (AB)\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u' \\ v' \end{pmatrix} = B\begin{pmatrix} u \\ v \end{pmatrix}$$

then

$$\begin{pmatrix} u'' \\ v'' \end{pmatrix} = A\left(B\begin{pmatrix} u \\ v \end{pmatrix}\right) = A\begin{pmatrix} u' \\ v' \end{pmatrix}$$

so one sees that $A^{(n)} B^{(n)} = (AB)^{(n)}$, hence this is a representation.

The rest of the statements are straightforward deductions from 1) and Prop. 6. QED

Let $z = x + yk \in H$.

Proposition 9

1) For any $i, j = 1, 2, \ldots, n+1$, the $(ij)^{th}$ entry $(\rho^n(z))_{ij}$ is a polynomial in $x$, $\bar{x}$, and $y$. with rational integer coefficients,

homogeneous of degree n.

2) The first row of $\rho^n(z)$ is given by

$$(\rho^n(z))_{1j} = \binom{n}{j-1} x^{n+1-j} y^{j-1} .$$

3) Let $x \in \mathbb{C}$. Then $\rho^n(x)$ is a diagonal matrix

$$\mathrm{diag}(x^{n+1-i} x^{i-1})_{i=1,2,\ldots,n+1} .$$

<u>Proof.</u> 1) Looking at the definition of the n-fold symmetric product, one sees that the entries of $A^{(n)}$ are polynomials in the entries of $A$ with rational integer coefficients. Homogeneity of degree n follows from Prop. 8.3.

The proofs of 2) and 3) also follow directly from the definition of the symmetric product. <u>QED</u>

The following two examples illustrate the typical form of the representations. In particular, notice how the "outside" entries are single terms, while one gets more complicated expressions as one approaches the "center" of the matrix. Let $x + yk \in H$.

$$\rho^{-2}(x + yk) = \left(\frac{1}{|x|^2 + y^2}\right)^2 \begin{pmatrix} \bar{x}^2 & -2\bar{x}y & y^2 \\ \bar{x}y & |x|^2 - y^2 & -xy \\ y^2 & 2xy & x^2 \end{pmatrix} .$$

$$\rho^{-4}(x + yk) = \left(\frac{1}{|x|^2 + y^2}\right)^4 \quad .$$

$$\begin{pmatrix} \bar{x}^4 & -4\bar{x}^3 y & 6\bar{x}^2 y^2 & -4\bar{x}y^3 & y^4 \\ \bar{x}^3 y & \bar{x}^2 |x|^2 - 3\bar{x}^2 y^2 & -3\bar{x}|x|^2 y + 3\bar{x}y^3 & -y^4 + 3|x|^2 y^2 & -xy^3 \\ \bar{x}^2 y^2 & 2\bar{x}|x|^2 y - 2\bar{x}y^3 & |x|^4 - 4|x|^2 y^2 + y^4 & -2|x|^2 xy + 2xy^3 & x^2 y^2 \\ \bar{x}y^3 & -y^4 + 3|x|^2 y^2 & -3xy^3 + 3|x|^2 xy & |x|^2 x^2 - 3x^2 y^2 & -x^3 y \\ y^4 & 4xy^3 & 6x^2 y^2 & 4x^3 y & x^4 \end{pmatrix}$$

## III. 3 Vector Modular Forms

We will define modular forms consistent with the model of Eisenstein series which we will develop in the next chapter. Let $\rho^n$ be the n-fold symmetric quaternion representation as before. Fix an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant = $d < 0$.

In order to concentrate on the properties of modular forms with class number $= h > 1$, we make some simplifying assumptions. I will assume that the discriminant $d \neq -3, -4$ and that the weight $n$ is even. These assumptions eliminate the need for multiplier systems and characters of the unit group. The reader interested in the more general case is urged to see

Hermann [ 5 ] or Patterson-Goldfeld [ 8 ] for an indication of the complications we are avoiding.

Notation. For the remainder of this thesis, we will be manipulating many matrices and vectors. To avoid specifying the subscripts each time, I will reserve the subscripts $i$ and $j$ to run through the set $\{1, 2, \ldots, n+1\}$. The only exception will be the few times when $i = \sqrt{-1}$ in which cases there should be no confusion. An $(n+1) \times (n+1)$ matrix $(c_{ij})_{i,j=1,2,\ldots,n+1}$ will be written $(c_{ij})$. An $(n+1)$-length column vector $(c_i)_{i=1,2,\ldots,n+1}$ will be denoted $(c_i)$. Similarly, I will reserve the subscript $m$ to run through the set $\{1, 2, \ldots, h\}$ where $h$ is the class number of the fixed field $K = \mathbb{Q}(\sqrt{d})$. Let $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_h$ be a fixed set of representatives of the ideal class group, with $\underline{a}_1 = (1)$. Then we will also drop the subscripts from h-length vectors subscripted by m. For instance, the h-length vector $(f(z, \underline{a}_m))_{m=1,2,\ldots,h}$ will be denoted $(f(z, \underline{a}_m))$ and $\left(A_{[\underline{a}_m]}\right)_{m=1,2,\ldots,h}$ will be written $\left(A_{[\underline{a}_m]}\right)$.

Let $[\underline{a}]$ be any ideal class. Let $f(z, \underline{a})$ be an $(n+1)$-length vector of complex-valued functions, say $(f_i(z, \underline{a}))$.

Definition. $f(z, \underline{a})$ is an $[\underline{a}]$ component of a vector modular form of weight $n$ if

1) each $f_i(z, \underline{a})$ is defined and continuous on H.

2) each $f_i(z, \underline{a})$ is bounded "near infinity," that is, on a subset of H given by $\{x + yk \mid y > 1\}$.

3) $f(A(z), \underline{a}) = \rho^n(\gamma z + \delta) f(z, \underline{a})$ for all $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_{\underline{a}}$.

Definition. The $[\underline{a}]$ component of the vector slash operator of weight $n$ is defined by

$$f(z, \underline{a}) \mid A = \rho^{-n}(\mu^{-1/2} (\gamma z + \delta)) f(\mu^{-1/2} A(z), \underline{a})$$

where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}}(\mu)$ for any $[\underline{b}]$, where one fixes either square root of $\mu \in O(Z)$.

Proposition 10. Let $A \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}}(\mu)$, and let $f(z, \underline{a})$ be an $[\underline{a}]$ component of a vector modular form.

1) Changing the choice of square root of $\mu$ does not change the slash operator.

2) $f(z, \underline{a}) \mid A$ is an $[\underline{a}\,\underline{b}^2 \mu^{-1}]$ component of a vector modular form. If $B \in \Gamma^{\underline{c}}_{\underline{c}\,\underline{a}\,\underline{b}^2 \mu^{-1}}(\nu)$, for any $[\underline{c}]$ and any $\nu \in O(Z)$, then $(f(z, \underline{a}) \mid A) \mid B = f(z, \underline{a}) \mid (AB)$.

Proof. 1) By assumption, $n$ is even so $\rho^n(-1) = \rho^n(1)$. Also $-\mu^{1/2} A(z) = \mu^{1/2} A(z)$ so replacing $\mu^{1/2}$ by $-\mu^{-1/2}$ does not change the slash operator.

2) This will follow from the consistency rule, Prop. 2.1,

and a careful application of Props. 6, 7 and 8. Let

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^{\underline{b}}_{\underline{b}\,\underline{a}}(\mu)$$

and

$$B = \begin{pmatrix} \pi & \rho \\ \psi & \tau \end{pmatrix} \in \Gamma^{\underline{c}}_{\underline{c}\,\underline{a}\,\underline{b}^2\mu^{-1}}(\nu) .$$

Then

$$f(z, \underline{a}) \mid A = \rho^{-n}(\mu^{-1/2}(\gamma z + \delta))\, f(\mu^{-1/2} A(z), \underline{a})$$

so

$$\begin{aligned}
(f(z,\underline{a}) \mid A) \mid B &= \rho^{-n}(\nu^{-1/2}(\psi z + \tau)) \cdot \\
&\quad \cdot\ \rho^{-n}(\mu^{-1/2}(\gamma[\nu^{-1/2} B(z)] + \delta))\, f(\mu^{-1/2} A \nu^{-1/2} B(z), \underline{a}) \\
&= \rho^{-n}(\nu^{-1/2}(\psi z + \tau)) \cdot \rho^{-n}(\mu^{-1/2}[\gamma \nu^{-1/2}(\pi z + \rho) \\
&\quad + \delta \nu^{-1/2}(\psi z + \tau)] \cdot (\psi z + \tau)^{-1} \nu^{1/2}) \cdot f((\mu\nu)^{-1/2} AB(z), \underline{a}) \\
&= \rho^{-n}((\mu\nu)^{-1/2}[(\gamma\pi + \delta\psi)z + (\gamma\rho + \delta\tau)]) \cdot \\
&\qquad \cdot\ f((\mu\nu)^{-1/2} AB(z), \underline{a}) \\
&= f(z, \underline{a}) \mid (AB) .
\end{aligned}$$

Notice that several times I have used 1) to replace $\mu^{1/2}\nu^{1/2}$ by $(\mu\nu)^{1/2}$. <u>QED</u>

<u>Definition.</u> $F(z)$ is a vector modular form of weight $n$, written $F \in \underline{F}(n)$, if $F$ is an h-length vector of functions $F(z) = (f(z, \underline{a}_m))$

where each $f(z, \underline{a}_m)$ is an $[\underline{a}_m]$ component of a vector modular form, $m = 1, 2, \ldots, h$.

Definition. The vector slash operator of weight $n$ is defined for $F \in \underline{F}(n)$ and $A = \left(A_{[\underline{a}_m]}\right) \in M(\mu, \underline{b})$ by

$$F(z) \mid A = (g(z, \underline{a}_m))$$

where

$$g(z, \underline{a}_m) = f(z, \underline{a}_m \underline{b}^{-2} \mu) \Big| A_{[\underline{a}_m \underline{b}^{-2} \mu]}$$

for $m = 1, 2, \ldots, h$.

Proposition 11

Let $F \in \underline{F}(n)$, $A \in M(\mu, \underline{b})$ and $B \in M(\nu, \underline{c})$ for any $\mu, \nu \in O(Z)$ and $[\underline{b}], [\underline{c}]$ ideal classes. Then $(F(z)|A)|B = F(z)|(AB)$. Furthermore, $F(z)|A = F(z)$ for $A \in M$.

Proof. By the consistency rule, Prop. 2.1, if $A = \left(A_{[\underline{a}_m]}\right)$ and $B = \left(B_{[\underline{a}_m]}\right)$, then $B_{[\underline{a}_m \underline{b}^2 \mu^{-1}]}$ acts on $f(z, \underline{a}_m) \Big| A_{[\underline{a}_m]}$ so the slash operator does send $\underline{F}(n)$ into $\underline{F}(n)$. The rest follows from the definitions and Prop. 10. QED

Let $\chi$ be a character of the ideal class group. Using Prop. 3, one sees that as a multiplicative semigroup, $M$ has index $h$ in $M(1, \underline{I})$. This suggests that $\underline{F}(n)$ can be decomposed into a direct sum of spaces $\underline{F}(n, \chi)$ depending on $\chi$.

Take $F \in \underline{F}(n)$, and set

$$F(z, \chi) = \frac{1}{h} \sum_{m=1}^{h} \overline{\chi}(\underline{a}_m) F(z) \mid C(\underline{a}_m) \quad .$$

Then Prop. 3.6 and simple calculation show that $F(z, \chi) \mid B = \chi(\underline{b}) F(z, \chi)$ for any $B \in M(1, \underline{b})$, for any $[\underline{b}]$. Define $\underline{F}(n, \chi)$ to be the set of all $F \in \underline{F}(n)$ such that $F(z) \mid B = \chi(\underline{b}) F(z)$ for every $B \in M(1, \underline{b})$ and all $[\underline{b}]$. The orthogonality relations for finite character sums show that $\underline{F}(n) = \bigoplus_{\chi} \underline{F}(n, \chi)$ where the direct sum is over all $\chi$ which are characters on $\underline{I}/\underline{P}$.

Proposition 12

Let $F \in \underline{F}(n, \chi)$ have $f(z, \underline{a}) = 0$ for some $[\underline{a}]$. Then $f(z, \underline{b}) = 0$ for all $[\underline{b}]$ such that $[\underline{a}\,\underline{b}]$ is a square in $\underline{I}/\underline{P}$. In particular, if $\underline{I}/\underline{P}$ has no 2-torsion, then $F = 0$.

Proof. $F \in \underline{F}(n, \chi)$ means that for any $[\underline{c}]$, $F(z) \mid C(\underline{c}) = \chi(\underline{c}) F(z)$. From the definition of the slash operator, this says that

$$f(z, \underline{a}_m \underline{c}^{-2}) \Big| C_{\frac{\underline{c}}{\underline{c}\,\underline{a}_m \underline{c}^{-2}}} = \chi(\underline{c}) f(z, \underline{a}_m) \quad .$$

Let $\underline{a}_m = \underline{a}\,\underline{c}^2$. Then we have that $0 = \chi(\underline{c}) f(z, \underline{a}\,\underline{c}^2)$ hence $0 = f(z, \underline{a}\,\underline{c}^2)$ for any $[\underline{c}]$. Now $[\underline{b}] = [\underline{a}\,\underline{c}^2]$ for some $[\underline{c}]$ iff $[\underline{a}\,\underline{b}]$ is a square in $\underline{I}/\underline{P}$. QED

Proposition 13

Let $F \in \underline{F}(n, \chi)$ with $F(z) = (f(z, \underline{a}_m))$. Then

$$f(z, \underline{a}_m) \,\Big|\, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \overline{\chi}(\underline{a}_m)\, f(z, \underline{a}_m^{-1}) \ .$$

Proof. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sim C^{\underline{a}_m^{-1}}_{\underline{a}_m^{-1}\underline{a}_m}$ and hence it follows from the definitions of the slash operator and $\underline{F}(n, \chi)$. QED

We now wish to decompose $\underline{F}(n, \chi)$ with respect to the units. Let $\varphi$ be a character on $U = \{1, -1\}$. There are only two characters, the trivial character and the character $\varphi(1) = 1$ and $\varphi(-1) = -1$. Given $F \in \underline{F}(n, \chi)$, define

$$F(z, \varphi) = \frac{1}{2}\left(F(z) + \varphi(-1)\, F(z) \,\Big|\, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right) \ .$$

Proposition 14

Take $F \in \underline{F}(n, \chi)$ and let $F(z, \varphi)$ be as above. Then $F(z, \varphi) | B = \chi(\underline{b})\, \varphi(-1)\, F(z, \varphi)$ for any $B \in M(-1, \underline{b})$ with any $[\underline{b}]$.

We get a decomposition $\underline{F}(n, \chi) = \bigoplus_{\varphi} F(n, \chi, \varphi)$ where $F \in F(n, \chi, \varphi)$ whenever

$$F(z) \,\Big|\, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \varphi(-1)\, F(z) \ .$$

<u>Proof.</u> $B \in M(-1, \underline{b})$ implies that $B = A \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ for some $A \in M(1, \underline{b})$. This gives the first statement, and then the orthogonality relations for finite character sums give the decomposition. <u>QED</u>

We want to extend any character $\varphi$ of $U$ to a character of $O(Z)$. Recall that the rational integers have two characters on the units, the trivial character and the signum character. The signum character extends to all rational integers by defining a subset of positive integers. Analogously, we will define "positive" elements of $O(Z)$. For each integral prime $(\pi)$, fix a generator $\pi \in O(Z)$, and call $\pi$ a "positive" prime element. Define $O_+(Z) = \{ \alpha \in O(Z) \mid \alpha = \pi_1^{P_1} \pi_2^{P_2} \cdots \pi_r^{P_r}$, with $P_s \in \mathbb{Z}$ and $\pi_s$ a positive prime element, $s = 1, 2, \ldots, r \}$. Let $O_+[\underline{a}_m] = O[\underline{a}_m] \cap O_+(Z)$.

<u>Proposition 15</u>

1) For any $\alpha \in O(Z)$, either $\alpha \in O_+(Z)$ or $-\alpha \in O_+(Z)$.

2) $\alpha^2 \in O_+(Z)$ for all $\alpha \in O(Z)$.

3) A character $\varphi$ of $U$ extends to $O(Z)$ by $\varphi(\mu) = 1$ for all $\mu \in O(Z)$ if $\varphi$ is the trivial character, and if $\varphi$ is not trivial then $\varphi(\mu) = 1$ for $\mu \in O_+(Z)$ and $\varphi(\mu) = -1$ for $-\mu \in O_+(Z)$.

Proof. Prop. 1.6 shows that one can get a prime factorization. In fact, this factorization is unique up to a unit since the prime factorization of the ideals is unique. The proof follows from this. QED

## III. 4 Renormalizing the Hecke Operators

In the classical case, one normalizes the Hecke operator $T(m)$ by multiplying by $m^{n/2-1}$ (see Ogg [7]). We make a similar normalization in our case.

Definition. The normalized Hecke operator $T(\mu)$ for $\mu \in O(Z)$ is defined by

$$T(\mu) = t(\mu) \cdot |\mu|^{n/2} N(\mu)^{-1}$$

where this means that one acts on each component of a vector modular form by $t(\mu)$ and then multiplies the resulting components by $|\mu|^{n/2} N(\mu)^{-1}$.

Recall that $n$ is even so $|\mu|^{n/2}$ is well-defined. We rephrase Theorems 1 and 2.

Theorem 1′

For any $\mu \in O(Z)$, $F \in \underline{F}(n, \chi)$ implies that $F(z)|T(\mu) \in \underline{F}(n, \chi)$.

Theorem 2′

For any $\mu, \nu \in O(Z)$, the Hecke operators satisfy the following relation on $\underline{F}(n, \chi)$:

$$T(\mu)\,T(\nu) = \sum_{\substack{(\theta)\ \text{integral} \\ \theta \mid (\mu,\nu)}} N(\theta)^{-1}\,\overline{\chi}(\theta)\,|\theta|^n\,T\left(\frac{\mu\nu}{\theta^2}\right).$$

Proof. First, $F(z) \left| \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \right. = F(z)$ and $F(z)\,|\,C(\theta^{-1}) = \overline{\chi}(\theta)\,F(z)$. Given these facts, the rest is straightforward calculation from Theorem 2 and the definitions. QED

Proposition 16

On $F(n,\chi,\varphi)$, $T(-\mu) = \varphi(-1)\,T(\mu)$.

Proof. $t(-\mu) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t(\mu)$ and $|\mu|^{\frac{n}{2}}\,N(\mu)^{-1}$ is invariant under a sign change. QED

## III. 5 Fourier Expansions

Let $F \in \underline{F}(n,\chi)$, $F(z) = (f(z,\underline{a}_m))$ and each $f(z,\underline{a}_m) = (f_i(z,\underline{a}_m))$. Now $f(z,\underline{a}_m) \left| \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \right. = f(z,\underline{a}_m)$ for all $\gamma \in O[\underline{a}_m^{-1}]$, so we expect to have a Fourier expansion.

Let $(\delta)$ be the different of $K/\mathbb{Q}$, where $\delta \in O_+(Z)$ is an ideal number associated to the different ideal. Let $\mathrm{tr}(x) = x + \overline{x} = 2\,\mathrm{Re}\,x$ for any $x \in \mathbb{C}$. By definition of the different, $(\delta)^{-1} = \delta^{-1} \cdot O(Z) \cap K$ is the dual lattice to $O(K)$ with respect to the trace. From this it follows that $\delta^{-1} \cdot O(Z) \cap K[\underline{a}_m]$ is the dual lattice to $O[\underline{a}_m^{-1}]$. Thus we expect a Fourier series for $f(z,\underline{a}_m)$

in terms of $\delta^{-1} \circ O(Z) \cap K[\underline{a}_m]$, that is, in terms of $e^{2\pi i \operatorname{tr}\left(\frac{\beta x}{\delta}\right)}$ for $\beta \in O[\delta \underline{a}_m]$. Set $e(\beta x/\delta) = e^{2\pi i \operatorname{tr}\left(\frac{\beta x}{\delta}\right)}$.

I will now restrict the possible coefficients of the Fourier expansion. Take $S$ to be a finite index set. For each $q \in S$, each $i = 1, 2, \ldots, n+1$, and each $m = 1, 2, \ldots, h$, let $c_i(\underline{a}_m, q)$ and $s(i, q)$ be complex numbers. These will correspond to the "constant" terms of the Fourier expansion. For any $\beta \in O(Z)$, define $W_i(y)$ to be a complex-valued function of a positive real variable, and let $c_i(\beta)$ be a complex number.

To guarantee convergence, we assume that $|c_i(\beta)| = O(|\beta|^{M_1})$ for some $M_1$ and that $|W_i(y)| = O(e^{-M_2 y})$ for some $M_2$. Here $O(-)$ means the usual big O bound. For convenience, if $W_i(y)$ is identically zero, then set all $c_i(\beta) = c_i(\underline{a}_m, q) = s(i, q) = 0$. Also assume that $s(i, q_1) = s(i, q_2)$ only if $q_1 = q_2$. Also let $W$ be the system of functions $\{y^{s(i,q)}; W_i(y) \mid q \in S, i = 1, 2, \ldots, n+1\}$.

<u>Definition</u>. $F \in \underline{F}(n, \chi)$ is said to be a W-system vector modular form, written $F \in \underline{F}(n, \chi, W)$, whenever each $f_i(z, \underline{a}_m)$ has a Fourier expansion

$$f_i(z, \underline{a}_m) = \sum_{q \in S} c_i(\underline{a}_m, q)\, y^{s(i,q)}$$

$$+ \sum_{\beta \in O[\delta \underline{a}_m]}' c_i(\beta)\, W_i(|\beta| y)\, e(\beta \dot{x}/\delta) \;,$$

where $\Sigma'$ means that one eliminates the term $\beta = 0$.

For convenience, we denote the Fourier expansion of $F \in \underline{F}(n, \chi, W)$ by $\{c_i(\underline{a}_m, q); c_i(\beta)\}$ where $m = 1, 2, \ldots, h$, $i = 1, 2, \ldots, n+1$, $q \in S$, and $\beta \in O(Z)$. One can verify that the ordinary theorems about uniqueness of Fourier series of functions periodic on a lattice say that the coefficient system $\{c_i(\underline{a}_m, q); c_i(\beta)\}$ uniquely determines $F \in \underline{F}(n, \chi, W)$.

Throughout the remainder of this section and the next two, we will deal exclusively with functions in $\underline{F}(n, \chi, W)$. Thus, throughout this and the two following sections, assume that any $F$ mentioned is in $\underline{F}(n, \chi, W)$, with $F(z) = (f(z, \underline{a}_m))$ where $f(z, \underline{a}_m) = (f_i(z, \underline{a}_m))$ and where $F$ has a Fourier coefficient system $\{c_i(\underline{a}_m, q); c_i(\beta)\}$. Similarly, assume that any $G$ mentioned is in $\underline{F}(n, \chi, W)$ with $G(z) = (g(z, \underline{a}_m))$ where $g(z, \underline{a}_m) = (g_i(z, \underline{a}_m))$ and where $G$ has a Fourier coefficient system $\{d_i(\underline{a}_m, q); d_i(\beta)\}$.

We now want to investigate the relationships between the $f_i(z, \underline{a}_m)$. First we need a technical lemma.

Lemma 3

Let $\{a_{\beta,i}\}$ be a set of complex numbers for all $\beta \in O[\delta \underline{a}_m]$ and $i = 1, 2, \ldots, n+1$. Assume that $\sum_\beta |a_{\beta,i}|$ converges for each i. Furthermore, assume that $\sum_\beta a_{\beta,i}\, e(\beta x/\delta)$ is periodic on $\mathbb{C}/O[\underline{a}_m^{-1}]$.

Then

$$\sum_{i=1}^{n+1} \sum_{\beta \in O[\delta \underline{a}_m]} a_{\beta,i}\, x^{i-1}\, e(\beta x/\delta) = 0$$

iff each $a_{\beta,i} = 0$.

Proof. We have

$$\sum_i \left( \sum_\beta a_{\beta,i}\, e\left(\frac{\beta x}{\delta}\right)\right) x^{i-1} = 0 \quad .$$

We may assume that there exists a $\beta$ such that $a_{\beta,n+1} \neq 0$ (otherwise replace n by n-1). Let B be a bound for all $\sum_\beta |a_{\beta,i}|$, so $B \geq \left| \sum_\beta a_{\beta,i}\, e(\beta x/\delta) \right|$ for all x. Now

$$x^n \sum_\beta a_{\beta,n+1}\, e(\beta x/\delta) = - \sum_{i=1}^{n} \left( \sum_\beta a_{\beta,i}\, e\left(\frac{\beta x}{\delta}\right)\right) x^{i-1} \quad .$$

Putting in absolute values, using the triangle inequality, and dividing by $|x|^n$, we get

$$\left|\sum_{\beta} a_{\beta,n+1}\, e\left(\frac{\beta x}{\delta}\right)\right| \le \sum_{i=1}^{n} |x|^{-n-1+i}\, B \;.$$

The general theory of Fourier series says that $\sum_{\beta} a_{\beta,n+1}\, e(\beta x/\delta) = 0$ for all $x \in \mathbb{C}$ iff all $a_{\beta,n+1} = 0$. By our assumption, therefore, there exists $x_0 \in \mathbb{C}$ such that $v = \sum a_{\beta,n+1}\, e(\beta x_0/\delta) \neq 0$. Now $v = \sum a_{\beta,n+1}\, e(\beta(x_0+p)/\delta)$ for any $p \in \mathbb{Z}$. Thus,

$$0 < |v| \le \sum_{i=1}^{n} |x_0 + p|^{-n-1+i}\, B$$

for every $p \in Z$, and thus $0 < |v| \le n\, p^{-1} \cdot B$ for all $p$ large, which is a contradiction, hence every $a_{\beta,i} = 0$. QED

Theorem 3

Take $F \in \underline{F}(n, \chi, W)$. Suppose that for some $m$, $f_1(z, \underline{a}_m) = 0$ for all $z \in H$. Then $f(z, \underline{a}_p) = 0$ for all $\underline{a}_p$ such that $[\underline{a}_m\, \underline{a}_p]$ is a square in $\underline{I}/\underline{P}$.

Proof. By Prop. 13,

$$\rho^{-n}(z)\,(f_i(-z^{-1}, \underline{a}_m)) = \overline{\chi}(\underline{a}_m)\,(f_i(z, \underline{a}_m^{-1}))$$

or thus

$$(f_i(-z^{-1}, \underline{a}_m)) = \overline{\chi}(\underline{a}_m)\, \rho^n(z)(f_i(z, \underline{a}_m^{-1})) \ .$$

By hypothesis $f_1(-z^{-1}, \underline{a}_m) = 0$ so using Prop. 1.2 we find that

$$0 = \sum_{i=1}^{n+1} \binom{n}{i-1} x^{n+1-i}\, y^{i-1}\, f_i(z, \underline{a}_m^{-1}) \ .$$

Now put in the Fourier expansions to get

$$0 = \sum_{i=1}^{n+1} x^{n+1-i} \binom{n}{i-1} y^{i-1} \left\{ \sum_{q \in S} c_i(\underline{a}_m^{-1}, q)\, y^{s(i,q)} \right.$$

$$\left. + \sum_{\beta \in O[\delta \underline{a}_m^{-1}]} c_i(\beta)\, W_i(|\beta|\, y)\, e(\beta x/\delta) \right\} \ .$$

We apply Lemma 3 to find that

$$\sum_{q \in S} c_i(\underline{a}_m^{-1}, q)\, y^{s(i,q)} = 0 \quad \text{and} \quad c_i(\beta) = 0 \ ,$$

hence $f(z, \underline{a}_m^{-1}) = 0$ for all $z \in H$. Prop. 12 finishes the proof. <u>QED</u>

<u>Remark</u>. If I had hypothesized that $f_i(z, \underline{a}_m) = 0$ for some other $i$, then the same proof would have worked, except that I would have used Prop. 9.1 instead of Prop. 9.2. This would mean that

instead of $c_i(\beta) = 0$, I would have obtained a sum of terms $y^r c_i(\beta) W_i(|\beta| y)$ for various $i$ and various powers $r$. Thus one would need to make some mild hypotheses regarding the linear independence of the $y^r W_i(y)$ functions.

## III. 6 Eigenforms

Definition. $F \in \underline{F}(n, \chi, W)$ is said to be an eigenform for $T(\mu)$ if there exists a constant $\lambda(\mu)$ such that $F(z) | T(\mu) = \lambda(\mu) F(z)$.

A natural question arises. In terms of the Fourier expansion, $F$ is an eigenform whenever $F(z) | T(\mu)$ has a Fourier coefficient system $\{\lambda(\mu) c_i(\underline{a}_m, q); \lambda(\mu) c_i(\beta)\}$. One might wonder why we do not consider constants $\lambda_i(\mu, \underline{a}_m)$ which depend on $i$ and $m$, so an eigenform would have a Fourier coefficient system $\{\lambda_i(\mu, \underline{a}_m) c_i(\underline{a}_m, q); \lambda_i(\mu, \beta) c_i(\beta)\}$.

Corollary to Theorem 3

Let $F$ have Fourier coefficient system $\{c_i(\underline{a}_m, q); c_i(\beta)\}$. If $\{\lambda_i(\mu, \underline{a}_m) c_i(\underline{a}_m, q); \lambda_i(\mu, \beta) c_i(\beta)\}$ corresponds to a vector modular form $F' \in \underline{F}(n, \chi, W)$, then $\lambda_i(\mu, \underline{a}) = \lambda_j(\mu, \underline{b})$ for all $i$ and $j$ and all $[\underline{a}\,\underline{b}]$ which are squares in $\underline{I}/\underline{P}$.

Proof. Let $G(z) = F'(z) - \lambda_1(\mu, \underline{a}) F(z)$. Then $g_1(z, \underline{a}) = 0$ hence we are done by Theorem 3. QED

Now we are ready to investigate the action of the Hecke operators on the Fourier expansion.

<u>Lemma 4</u>

1) Let $\kappa\lambda = \mu$ and $z = x + yk \in H$. Then

$$\mu^{-1/2} \begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix} = \frac{\kappa}{\lambda} x + \frac{\sigma}{\lambda} + \left|\frac{\kappa}{\lambda}\right| yk \ .$$

2) Let $(g(z, \underline{a}_m)) = F(z) \left| \left( \begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix} \right) \right.$ for $\sigma \in O[\lambda]$.

Then

$$g(z, \underline{a}_m \kappa^2 \mu^{-1}) = \rho^{-n}(\mu^{-1/2}\lambda)$$

$$\cdot \left\{ \left( \sum_q c_i \left( \underline{a}_m \frac{\mu}{\kappa^2}, q \right) \left| \frac{\kappa^2}{\mu} \right|^{s(i,q)} y^{s(i,q)} \right) \right.$$

$$+ \sum_{\beta \in O[\delta \underline{a}_m]} \left( c_i(\beta)\, W_i\left( \left| \frac{\beta\kappa^2}{\mu} \right| y \right) \right)_{i=1,2,\ldots,n+1} e\left( \frac{\beta\sigma}{\delta\lambda} \right)$$

$$\left. \cdot\ e\left( \frac{\beta\kappa^2}{\delta\mu} x \right) \right\} \ .$$

<u>Proof.</u> Prop. 6.3 and the definitions gives 1). The definition of the vector slash operator and straightforward calculation yields 2). <u>QED</u>

Theorem 4

For any $\mu \in O(z)$, $F(z)|T(\mu) \in \underline{F}(n, \chi, W)$ and has a Fourier coefficient system $\{d_i(\underline{a}_m, q); d_i(\gamma)\}$ where

$$(d_i(\underline{a}_m, q)) = \sum_{\substack{(\kappa) \\ \kappa | \mu}} \overline{\chi}(\kappa)\, \rho^n(\kappa)\, N(\kappa)^{-1}\, \rho^{-n}\left(\left(\frac{\mu}{|\mu|}\right)^{1/2}\right)$$

$$\cdot \left(c_i\left(\underline{a}_m \frac{\mu}{\kappa^2}\right)\left|\frac{\kappa^2}{\mu}\right|^{s(i,q)}\right)$$

and where

$$(d_i(\gamma)) = \sum_{\substack{(\kappa) \\ \kappa | (\gamma, \mu)}} \overline{\chi}(\kappa)\, N(\kappa)^{-1}\, \rho^n(\kappa)\, \rho^{-n}\left(\left(\frac{\mu}{|\mu|}\right)^{1/2}\right)\left(c_i\left(\frac{\gamma\mu}{\kappa^2}\right)\right).$$

Here $\sum_{\substack{(\kappa) \\ \kappa|\mu}}$ means sum over all ideals $(\kappa)$ where we choose some generator $\kappa$. Notice also that the "vector convention" is being used, so the above are identities of (n+1)-length vectors.

**Proof.** $F(z)|T(\mu) =$

$$F(z) \left| \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} C(\kappa^{-1}) \sum_{\substack{\sigma \bmod \lambda \\ \sigma \in O[\lambda]}} \left(\begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix}\right) \cdot |\mu|^{n/2} N(\mu)^{-1} \right.$$

$$= \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} \overline{\chi}(\kappa) \sum_{\substack{\sigma \bmod \lambda \\ \sigma \in O[\lambda]}} F(z) \left| \left(\begin{pmatrix} \kappa & \sigma \\ 0 & \lambda \end{pmatrix}\right) |\mu|^{n/2} N(\mu)^{-1} \right.$$

Let $(g(z, \underline{a}_m)) = F(z)|T(\mu)$. Lemma 4.2 and Prop. 8.3 show that

$$g(z, \underline{a}_m) = \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} \overline{\chi}(\kappa)\, \rho^{-n}\left(\left(\frac{\mu}{|\mu|}\right)^{1/2} \kappa^{-1}\right) N(\mu)^{-1}$$

$$\cdot \left\{ \sum_{\substack{\sigma \bmod \lambda \\ \sigma \in O[\lambda]}} \left( \sum c_i\left(\underline{a}_m \frac{\mu}{\kappa^2}, q\right) \left|\frac{\kappa^2}{\mu}\right|^{s(i,q)} y^{s(i,q)} \right) \right.$$

$$+ \sum_{\beta \in O\left[\delta \underline{a}_m \frac{\mu}{\kappa^2}\right]}{}' \left( c_i(\beta)\, W_i\left(\left|\frac{\beta\kappa^2}{\mu}\right| y\right)\right)$$

$$\left. \cdot \sum_{\substack{\sigma \bmod \lambda \\ \sigma \in O[\lambda]}} e\left(\frac{\beta\, \alpha_{k(m)}\, \sigma}{\delta\lambda}\right) \cdot e\left(\frac{\beta\kappa^2}{\delta\mu} x\right) \right\}$$

The properties of finite character sums show that

$$\sum_{\sigma \bmod \lambda} e\left(\frac{\beta\, \alpha_{k(m)}\, \sigma}{\delta\lambda}\right) = 0 \quad \text{if} \quad \lambda \nmid \beta$$

$$= N(\lambda) \quad \text{if} \quad \lambda \mid \beta \ .$$

Let $\gamma = \dfrac{\beta\kappa^2}{\mu}$. Then $\lambda \mid \beta$ iff $\kappa \mid \gamma$. Thus,

$$g(z, \underline{a}_m) = \sum_{\substack{(\kappa) \\ \kappa\lambda = \mu}} \overline{\chi}(\kappa)\, \rho^{-n}\left(\left(\frac{\mu}{|\mu|}\right)^{1/2}\right) N(\kappa)^{-1}\, \rho^{n}(\kappa)$$

$$\cdot \left\{ \left( \sum_{q} \left( c_i \;\; \underline{a}_m \frac{\mu}{\kappa^2},\; q \right) \left| \frac{\kappa^2}{\mu} \right|^{s(i,q)} y^{s(i,q)} \right) \right.$$

$$\left. + \sum_{\substack{\gamma \in O[\delta \underline{a}_m] \\ \kappa \mid \gamma}} \left( c_i \left( \frac{\gamma \mu}{\kappa^2} \right) W_i(|\gamma| y) \right) e(\gamma x/\delta) \right\} .$$

The diagonal matrices $\rho^{-n}\left(\left(\frac{\mu}{|\mu|}\right)^{1/2}\right)$ and $\rho^{n}(\kappa)$ do not permute the $W_i(|\gamma| y)$ terms, so we see that $F(z)|T(\mu)$ has the desired Fourier coefficient system. <u>QED</u>

<u>Corollary to Theorem 4</u> (hereafter called Cor. 4)

Suppose that $F(z)|T(\mu) = \lambda(\mu) F(z)$ and that $F(z)|T(\nu) = \lambda(\nu) F(z)$. Then

1) $$\rho^{n}\left(\left(\frac{\mu}{|\mu|}\right)^{1/2}\right)(\lambda(\mu)\, c_i(\nu)) = \rho^{n}\left(\left(\frac{\nu}{|\nu|}\right)^{1/2}\right)(\lambda(\nu)\, c_i(\mu)) .$$

2) $$\lambda(\mu)\, c_i(1) = \left(\frac{\overline{\mu}}{|\mu|}\right)^{\frac{1}{2}(n+1-i)} \left(\frac{\mu}{|\mu|}\right)^{\frac{1}{2}(i-1)} c_i(\mu) .$$

<u>Proof.</u> 1) Theorem 4 implies that

$$\rho^{n}\left(\left(\frac{\mu}{|\mu|}\right)^{1/2}\right)(\lambda(\mu)\, c_i(\nu)) = \sum_{\substack{(\kappa) \\ \kappa \mid (\mu, \nu)}} \overline{\chi}(\kappa)\, N(\kappa)^{-1} \rho^{n}(\kappa) \left( c_i \left( \frac{\mu\nu}{\kappa^2} \right) \right) .$$

The right is symmetric in $\mu$ and $\nu$, hence we have 1). To get 2), set $\nu = 1$ and use Prop. 9.3. <u>QED</u>

<u>Definition.</u> $F$ is said to be a normalized vector form if $c_1(1) = 1$.

<u>Proposition 17</u>

1) Suppose that $F$ is a non-zero simultaneous eigenform for all $T(\mu)$. Then $c_1(1) \neq 0$.

2) Suppose that $F$ and $F'$ are normalized eigenforms for all $T(\mu)$ with the same eigenvalues. Then $F = F'$.

<u>Proof</u>

1) Suppose $c_1(1) = 0$. Then by Cor. 4.2, $c_1(\mu) = 0$ for every $\mu$, hence one can show that $f_1(z, \underline{a}_m) = 0$ for every $m$. By Theorem 3, $F = 0$.

2) Since they have the same eigenvalues, $F - F'$ is also a simultaneous eigenform. The corresponding $c_1(1)$ term for $F - F'$ is $0$, hence we are done by 1). <u>QED</u>

<u>Proposition 18</u>

1) Let $\pi$ be prime. Choose $\sigma \in O[\pi]$ such that $\pi \nmid \sigma$. Then given any $[\underline{a}]$, there exists $A \in M(1, \underline{a})$ and $B \in M(1, \underline{a}^{-1})$ such that $\left(\begin{pmatrix} \pi & \sigma \\ 0 & \pi \end{pmatrix}\right) = A \left(\begin{pmatrix} 1 & 0 \\ 0 & \pi^2 \end{pmatrix}\right) B$.

2) Suppose that $F \in \underline{F}(n, \chi, W)$ has $c_1(\beta) = 0$ whenever $\pi \nmid \beta$. Then $F = 0$.

<u>Proof.</u> 1) We need to show that there exists $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ of determinant 1 such that

<u>Theorem 5</u>

1) Let $\pi$ be prime and $F \in \underline{F}(n, \chi, W)$. Suppose that there exists $\lambda(\pi)$ such that

$$\lambda(\pi)\, c_1(1) = \left(\frac{\pi}{|\pi|}\right)^{n/2} c_1(\pi)$$

and also that $c_1(1)\, c_1(\beta\pi) = c_1(\beta)\, c_1(\pi)$ for all $\beta$ with $\pi \nmid \beta$. Then $F(z)|T(\pi) = \lambda(\pi)\, F(z)$.

2) Suppose that $c_1(1)\, c_1(\mu\nu) = c_1(\mu)\, c_1(\nu)$ for all $(\mu, \nu) = 1$. Then $F$ is a simultaneous eigenform for all $T(\mu)$.

<u>Proof</u>. 1) Let $G(z) = F(z)|T(\pi) - \lambda(\pi)\, F(z)$. Then our hypotheses along with Theorem 4 show that $d_1(\beta) = 0$ whenever $\pi \nmid \beta$. Prop. 18.2 says that $G = 0$, hence our result.

2) Theorem 2′ says that the prime Hecke operators generate the entire Hecke algebra, so it is sufficient to show that $F$ is an eigenform for $T(\pi)$ with $\pi$ prime. If $c_1(1) = 0$, then by hypothesis $c_1(\mu) = 0$ for $\mu$ and hence by Theorem 3, $F = 0$. Therefore, assume $c_1(1) \neq 0$. For each prime $\pi$, let

$$\lambda(\pi) = \left(\frac{\bar{\mu}}{|\mu|}\right)^{n/2} \frac{c_1(\pi)}{c_1(1)} .$$

Then $G(z) = F(z)|T(\pi) - \lambda(\pi)\, F(z)$ has $d_1(\beta) = 0$ whenever $\pi \nmid \beta$. Prop. 18.2 says that $G = 0$, hence our result. <u>QED</u>

$$\begin{pmatrix} \pi & \alpha_m \sigma \\ 0 & \pi \end{pmatrix} \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi^2 \end{pmatrix}$$

where the class of any one element, say $[\gamma']$, is fixed.

Choose $\gamma'$ in any desired ideal class such that $\pi \nmid \gamma'$. Choose $\delta$ such that $\gamma' \alpha_m \sigma \delta \equiv 1 \bmod \pi$ which is possible by Prop. 1.3. By multiplying the matrices above, one sees that this determines $\delta' = \pi\delta$ and $\gamma = \pi\gamma'$. By Prop. 1.2 we can find $\alpha'$ and $\beta'$ such that $\alpha'\delta' - \beta'\gamma' = 1$. Now $\pi \mid \delta'$ so $\beta'\gamma' \equiv -1 \bmod \pi$ hence $\pi \mid \beta' + \delta\alpha_m\sigma$. Let $\beta = (\beta' + \delta\alpha_m\sigma)/\pi$ and $\alpha = \pi\alpha' + \gamma'\alpha_m\sigma$. Direct computation shows that $\alpha\delta - \beta\gamma = 1$ and that the matrix equation is satisfied. Prop. 3.3 now finishes the proof of 1).

2) Let $G(z) = F(z) \left| \left(\begin{pmatrix} \pi & \sigma \\ 0 & \pi \end{pmatrix}\right) \right. - F(z)$. Lemma 4.2 and the hypotheses imply that $g_1(z, \underline{a}_m) = 0$ for all $m$, hence by Theorem 3, $G = 0$. Take $\sigma \in O[\pi]$ with $\pi \nmid \sigma$. Then by 1), $F(z) \left| \left(\begin{pmatrix} 1 & 0 \\ 0 & \pi^2 \end{pmatrix}\right) \right. = F(z)$. Lemma 4.2 gives the Fourier coefficients for the left side, and equating coefficients yields the recursion relation

$$(c_i(\beta)) = \rho^n(\pi)\,(c_i(\beta/\pi^2))$$

where $c_i(\beta/\pi^2) = 0$ if $\pi^2 \nmid \beta$. Thus $c_i(\beta) = 0$ for all $\beta$, hence one can show that $F = 0$. <u>QED</u>

## III. 7 Dirichlet Series

The Dirichlet series are generally derived via the Mellin transform or via the eigenvalues of Hecke operators. We will first find Dirichlet series via Mellin transforms and then show their connection to the ones found via Hecke theory.

Definition. Let $F \in \underline{F}(n, \chi, W)$ have Fourier coefficient system $\{c_i(\underline{a}_m, q); c_i(\beta)\}$. For each $m$ and each $i$, define a Dirichlet series

$$D_i(s, \underline{a}_m) = \sum_{\beta \in O[\delta \underline{a}_m]}{}' \frac{c_i(\beta)}{|\beta|^s}$$

where $\Sigma'$ means that we eliminate $\beta = 0$. Set

$$V_i(s) = \int_0^\infty y^s \, W_i(y) \, \frac{dy}{y} .$$

Theorem 5

The Dirichlet series associated to $F \in \underline{F}(n, \chi, W)$ satisfy the functional equations

$$(D_i(s, \underline{a}_m) V_i(s)) = \rho^n(k) \, \overline{\chi}(\underline{a}_m) \, (D_i(n-s, \underline{a}_m) \, V_i(n-s)) .$$

<u>Proof.</u> For convenience, let the vector

$$c(y, \underline{a}_m) = \left( \sum_{q \in S} c_i(\underline{a}_m, q)\, y^{s(i,q)} \right)$$

be the "constant term" of $f(z, \underline{a}_m)$. Recall that $W_i(y)$ dies exponentially as $y$ goes to infinity, and that the $c_i(\beta)$ are polynomially bounded. This is used implicitly to justify convergence in the integrals below. Recall that $k$ is the quaternionic basis element.

$$\int_0^\infty y^s [f(ky, \underline{a}_m) - c(y, \underline{a}_m)] \frac{dy}{y}$$

converges for $\mathrm{Re}(s)$ large. Break the integral into $\int_0^1 + \int_1^\infty$ and in the first integral replace $y$ by $\frac{1}{y}$. Then we get

$$\int_1^\infty y^{-s} \left[ f(k/y, \underline{a}_m) - c\left(\frac{1}{y}, \underline{a}_m\right)\right] \frac{dy}{y}$$

$$+ \int_1^\infty y^s [f(ky, \underline{a}_m) - c(y, \underline{a}_m)] \frac{dy}{y} .$$

Prop. 18 says that $f(k/y, \underline{a}_m) = \rho^n(yk)\, \overline{\chi}(\underline{a}_m)\, f(yk, \underline{a}_m^{-1})$.
Prop. 8.3 says that $\rho^n(yk) = y^n \rho^n(k)$. Thus the first integral

becomes

$$\int_1^\infty y^{-s}\left[y^n \rho^n(k)\,\overline{\chi}(\underline{a}_m)\, f(yk, \underline{a}_m^{-1}) - c\left(\frac{1}{y}, \underline{a}_m\right)\right]\frac{dy}{y}$$

$$= \int_1^\infty y^{n-s}\,\rho^n(k)\,\overline{\chi}(\underline{a}_m)\,[f(yk, \underline{a}_m^{-1}) - c(y, \underline{a}_m^{-1})]\frac{dy}{y}$$

$$+ \int_1^\infty y^{-s}\left[y^n \rho^n(k)\,\overline{\chi}(\underline{a}_m)\, c(y, \underline{a}_m^{-1}) - c\left(\frac{1}{y}, \underline{a}_m\right)\right]\frac{dy}{y} \; .$$

Notice that the first integral converges for all $s$ and that the second integral converges for $\mathrm{Re}(s)$ large. Now we use the specific form of the constant terms. For any $p < 0$,

$$\int_1^\infty y^p \frac{dy}{y} = \frac{-1}{p} \; .$$

Thus for $\mathrm{Re}(s)$ large, the second integral can be evaluated to be

$$\rho^n(k)\,\overline{\chi}(\underline{a}_m)\left(\sum_{q \in S} c_i(\underline{a}_m^{-1}, q)\,\frac{-1}{n - s + s(i,q)}\right)$$

$$+ \left(\sum_{q \in S} c_i(\underline{a}_m, q)\,\frac{-1}{s + s(i,q)}\right) \; .$$

There is an obvious analytic continuation to all $s \in C$ with only a finite number of poles. Thus, all of the integrals above have analytic continuations to the entire plane except for these finite number of poles. Thus we can conclude that when interpreted via the analytic continuations,

$$\int_0^\infty y^s\,[f(yk, \underline{a}_m) - c(y, \underline{a}_m)]\,\frac{dy}{y} = \rho^n(k)\,\overline{\chi}(\underline{a}_m)$$

$$\cdot \int_0^\infty y^{n-s}\,[f(yk, \underline{a}_m^{-1}) - c(y, \underline{a}_m^{-1})]\,\frac{dy}{y}\ .$$

Now we put in the Fourier expansions to get

$$\int_0^\infty y^s \left[\sum_{\beta \in O[\delta \underline{a}_m]}{}' (c_i(\beta)\, W_i(|\beta| y))\right] \frac{dy}{y} = \rho^n(k)\,\overline{\chi}(\underline{a}_m)$$

$$\cdot \int_0^\infty y^{n-s} \left[\sum_{\beta \in O[\delta \underline{a}_m^{-1}]}{}' (c_i(\beta)\, W_i(|\beta| y))\right] \frac{dy}{y}\,.$$

Consider the integral on the left side. For $\mathrm{Re}(s)$ large, everything is absolutely convergent and if we replace $y$ by $\frac{y}{|\beta|}$ we get that the left hand side becomes

$$\left(\sum_{\beta \in O[\delta \underline{a}_m]} \frac{c_i(\beta)}{|\beta|^s} \int_0^\infty y^s\, W_i(y)\,\frac{dy}{y}\right) = (D_i(s, \underline{a}_m)\, V_i(s))\quad .$$

As we showed above, this can be analytically continued to all $s$.

Consider now the integral on the right side. For $\mathrm{Re}(s)$ large negative, everything is absolutely convergent so one gets $(D_i(n-s, \underline{a}_m^{-1})\, V_i(n-s))$ which also has an analytic continuation. Putting everything together, we get our result. <u>QED</u>

Now we use Theorem 2′ to define Dirichlet series via Hecke theory. Let $F \in F(n, \chi, W, \varphi)$ be an eigenform for all $T(\mu)$ with eigenvalues $\lambda(\mu)$. Take any strictly multiplicative function $v$ on $O(Z)$. Define

$$\Delta(s, v) = \sum_{\mu \in O_+(Z)} \frac{\lambda(\mu)\, v(\mu)}{|\mu|^s} .$$

By Theorem 2′,

$$\Delta(s, v) = \prod_{\substack{\pi \in O_+(Z) \\ \pi \text{ prime}}} \left(1 - \frac{\lambda(\pi)\, v(\pi)}{|\pi|^s} + \frac{N(\pi)^{-1}\, \overline{\chi}(\pi)\, v(\pi^2)}{|\pi|^{2s-n}}\right)^{-1} .$$

When $F \in \underline{F}(n, \chi, W, \varphi)$, Prop. 16 shows that $\lambda(-\mu) = \varphi(-1)\, \lambda(\mu)$. If $-\mu \in O_+(Z)$, then $\lambda(-\mu) = \varphi(\mu)\, \lambda(\mu)$. Thus,

$$\Delta(s, v) = \frac{1}{2} \sum_{\mu \in O(Z)} \frac{\lambda(\mu)\, \varphi(\mu)\, v(\mu)}{|\mu|^s}$$

$$= \prod_{\substack{(\pi) \\ \pi \text{ prime}}} \left(1 - \frac{\lambda(\pi)\, \varphi(\pi)\, v(\pi)}{|\pi|^s} + \frac{N(\pi)^{-1}\, \overline{\chi}(\pi)\, v(\pi^2)}{|\pi|^{2s-n}}\right)^{-1} .$$

We will now introduce a particular $v$ which will connect the $\Delta(s, v)$ Dirichlet series with the $D_i(s, \underline{a}_m)$ Dirichlet series. Let $\psi$ be any character on $\underline{I}/\underline{P}$. Define

$$\psi_i(\mu) = \varphi(\mu)\,\psi(\mu)\left(\frac{\mu}{|\mu|}\right)^{\frac{1}{2}(n+1-i)}\left(\frac{\overline{\mu}}{|\mu|}\right)^{\frac{1}{2}(i-1)} .$$

Then $\psi_i$ is a totally multiplicative function. Recall that under very mild assumptions on the W-system, we have already remarked that all $c_i(1) \neq 0$, and in fact if $c_i(1) = 0$ then we can assume that $\lambda(\mu) = 0$. Thus, assume that $c_i(1) \neq 0$ and use Cor. 4.2 to obtain

$$\Delta(s, \psi_i) = \frac{1}{2\,c_i(1)} \sum_{\mu \in O(Z)} \frac{c_i(\mu)\,\psi(\mu)}{|\mu|^s}$$

$$= \frac{\psi(\delta)}{2\,c_i(1)} \sum_{m=1}^{h} \psi(\underline{a}_m)\, D_i(s, \underline{a}_m) .$$

One can now verify the functional equation

$$(c_i(1)\,\overline{\psi}(\delta)\,\Delta(s, \psi_i)\,V_i(s)) = \rho^n(k)$$

$$\cdot\ (c_i(1)\,(\overline{\chi}\psi)(\delta)\ \Delta(n-s,\ (\chi\overline{\psi})_i)\,V_i(n-s))$$

where $(\chi\overline{\psi})(\mu) = \chi(\mu)\,\overline{\psi}(\mu)$. We also get an Euler product

$$\Delta(s, \psi_i) = \prod_{(\pi)\text{ prime}}$$

$$\cdot \left(1 - \frac{c_i(\pi)\,\psi(\pi)}{c_i(1)\ |\pi|^s} + \frac{N(\pi)^{-1}\,\overline{\chi}(\pi)\,\psi(\pi^2)}{|\pi|^{2s-n}}\left(\frac{\pi}{|\pi|}\right)^{n+2-2i}\right)^{-1}$$

III. 8 Principal Theorem

One might expect that $\underline{F}(n, \chi, W)$ has a basis of simultaneous vector eigenforms of the Hecke operators. I will not show this, but the interested reader is urged to see Stark [10] for a discussion of the needed fundamental domains, and then Hermann [5] for an indication of how to form the Petersson inner product. Assuming this, however, it is reasonable to ask if the principal component $f(z, \underline{a}_1)$ of a simultaneous eigenform $F(z)$ determines the entire vector form. The following theorem will give sufficient conditions for this to be true.

Recall the notation of Chapter 2, Section 1. Let $\beta_1, \ldots, \beta_r$ be a basis for the ideal class group, with $N_s$ equal the order of $[\beta_s]$. As shown by Dirichlet, we may assume that each $\beta_s$ is prime. This is convenient but not necessary; I actually only require that they be relatively prime.

By Theorem 2′, the prime Hecke operators generate the entire Hecke algebra. Recall that the Hecke operator $T(\pi)$ is actually an h-length vector of matrix operators $T_{[\underline{a}_m]}(\mu)$. If $\pi_1$ and $\pi_2$ are two primes, then $F(z)|T(\pi_1)\, T(\pi_2)$ is actually the vector form

$$\left(f(z, \underline{a}_m \pi_1 \pi_2) \,\Big|\, T_{[\underline{a}_m \pi_1 \pi_2]}(\pi_1)\, T_{[\underline{a}_m \pi_2]}(\pi_2)\right) .$$

By Cor. 2, $T(\pi_1)\, T(\pi_2) = T(\pi_2)\, T(\pi_1)$ so one concludes that

$$f(z, \underline{a}_m \pi_1 \pi_2) \,\Big|\, T_{[\underline{a}_m \pi_1 \pi_2]}(\pi_1)\, T_{[\underline{a}_m \pi_2]}(\pi_2)$$

$$= f(z, \underline{a}_m \pi_1 \pi_2) \,\Big|\, T_{[\underline{a}_m \pi_1 \pi_2]}(\pi_2)\, T_{[\underline{a}_m \pi_1]}(\pi_1)$$

for each $m$. Notice that one can easily determine the appropriate indices for the $T(\pi)$ components from the definition of the slash operator. Thus it should not be confusing to adopt the following simplifying notation: define $f(z, \underline{a}_m)|T(\pi_1)\, T(\pi_2)$ to be $f(z, \underline{a}_m)\Big|T_{[\underline{a}_m]}(\pi_1)\, T_{[\underline{a}_m \pi_1^{-1}]}(\pi_2)$. Let $f(z, \underline{a}_m)\Big|T(\pi)^r$ be defined by

$$f(z, \underline{a}_m)\Big|T_{[\underline{a}_m]}(\pi)\, T_{[\underline{a}_m \pi^{-1}]}(\pi) \cdots T_{[\underline{a}_m \pi^{r-1}]}(\pi) .$$

In other words, we will drop all of the subscripts from the components of the Hecke operators. It should be clear to the reader whether $T(\pi)$ refers to the vector Hecke operator or to a component of the Hecke operator. Notice that

$$f(z, \underline{a}_m) | T(\pi_1)\, T(\pi_2) = f(z, \underline{a}_m) | T(\pi_2)\, T(\pi_1)$$

according to our discussion above.

Theorem 6

Let $f(z, \underline{a}_1)$ be an (n+1)-length vector of complex valued functions for $z \in H$. Assume that $f$ satisfies the following hypotheses.

1) $f(z, \underline{a}_1) | A = f(z, \underline{a}_1)$ for all $A \in SL(2, O(K))$.

2) For each $s = 1, 2, \ldots, r$, there exists $L(\beta_s) \neq 0$ such that $f(z, \underline{a}_1) \Big| T(\beta_s)^{N_s} = L(\beta_s)\, f(z, \underline{a}_1)$.

3) If $\pi$ is prime such that $\left[\pi \beta_1^{P_1} \beta_2^{P_2} \cdots \beta_r^{P_r}\right]$ is principal, then there exists $L(\pi)$ such that

$$f(z, \underline{a}_1) \Big| T(\beta_1)^{P_1}\, T(\beta_2)^{P_2} \cdots T(\beta_r)^{P_r}\, T(\pi) = L(\pi)\, f(z, \underline{a}_1)\,.$$

Then there exists a vector eigenform $F \in \underline{F}(n)$ with first component $f(z, \underline{a}_1)$. Furthermore, if $F \in \underline{F}(n, \chi)$ and if $f(z, \underline{a}_1)$ has a Fourier expansion in terms of a W-system, then

$F \in \underline{F}(n, \chi, W)$.

Proof. Let $\lambda(\beta_s) = L(\beta_s)^{1/N_s}$ where we fix some branch of the $N^{th}$ root, with $N = \ell.c.m.(N_1, N_2, \ldots, N_r)$. Define

$$f(z; P_1, P_2, \ldots, P_r) = \prod_{s=1}^{r} \lambda(\beta_s)^{-P_s} f(z, \underline{a}_1) \Big| T(\beta_1)^{P_1} \cdots T(\beta_r)^{P_r}.$$

Suppose that we replace any $P_t$ by $P_t' = P_t + pN_t$, where $p \in \mathbb{Z}$. Then using commutativity,

$$f(z; P_1, \ldots, P_t', \ldots, P_r) = \left( \prod_{s=1}^{r} \lambda(\beta_s)^{-P_s} \right) \cdot \lambda(\beta_t)^{-pN_t}$$

$$\cdot\, f(z, \underline{a}_1) \Big| T(\beta_t)^{pN_t} \cdot T(\beta_1)^{P_1} \cdots T(\beta_r)^{P_r}$$

$$= f(z; P_1, P_2, \ldots, P_r),$$

using hypothesis 2). We conclude that $f(z; P_1, \ldots, P_r)$ is well-defined for $P_s \bmod N_s$, $s = 1, 2, \ldots, r$. Set $f(z, \underline{a}_m) = f(z; P_1, P_2, \ldots, P_r)$ where $[\underline{a}_m^{-1}] = \left[ \beta_1^{P_1} \cdots \beta_r^{P_r} \right]$.

Define $F(z) = (f(z, \underline{a}_m))$. I claim that $F$ is the desired eigenform. One can show using Prop. 2.1 and the definition of vector matrix multiplication that $F(z) | A = F(z)$ for all $A \in M$. Thus $F \in \underline{F}(n)$. Now for any $t = 1, 2, \ldots, r$,

$$f(z, \underline{a}_m) | T(\beta_t) = f(z; P_1, \ldots, P_r) | T(\beta_t)$$

$$= \prod_{s=1}^{r} \lambda(\beta_s)^{-P_s} f(z, \underline{a}_1) \Big| T(\beta_1)^{P_1} \cdots T(\beta_t)^{P_t+1} \cdots T(\beta_r)^{P_r}$$

$$= \lambda(\beta_t) f(z; P_1, \ldots, P_t+1, \ldots, P_r)$$

$$= \lambda(\beta_t) f(z, \underline{a}_m \beta_t^{-1}) .$$

Thus $F(z)|T(\beta_t) = \lambda(\beta_t) F(z)$. Suppose $\pi$ is prime with $\left[\pi\beta_1^{Q_1} \cdots \beta_r^{Q_r}\right]$ principal. Then

$$f(z, \underline{a}_m) \Big| T(\pi) = \prod_{s=1}^{r} \lambda(\beta_s)^{-P_s} f(z, \underline{a}_1) \Big| T(\beta_1)^{P_1} \cdots T(\beta_r)^{P_r} T(\pi)$$

$$= \prod_{s=1}^{r} \lambda(\beta_s)^{-P_s+Q_s} \cdot \prod_{s=1}^{r} \lambda(\beta_s)^{-Q_s} f(z, \underline{a}_1) \Big|$$

$$\cdot T(\beta_1)^{Q_1} \cdots T(\beta_r)^{Q_r} T(\pi) T(\beta_1)^{P_1-Q_1} \cdots T(\beta_r)^{P_r-Q_r}$$

$$= \lambda(\pi) \cdot \prod_{s=1}^{r} \lambda(\beta_s)^{-P_s+Q_s} f(z, \underline{a}_1) \Big| T(\beta_1)^{P_1-Q_1} \cdots T(\beta_r)^{P_r-Q_r}$$

$$= \lambda(\pi) f(z, \underline{a}_m \pi^{-1})$$

where

$$\lambda(\pi) = L(\pi) \prod_{s=1}^{r} \lambda(\beta_s)^{-Q_s} .$$

Thus, $F(z)|T(\pi) = \lambda(\pi) F(z)$, for all prime $\pi$, hence $F$ is a simultaneous eigenform for all the Hecke operators.

Suppose that $F \in \underline{F}(n, \chi)$ and $f(z, \underline{a}_1)$ has a W-system $\{y^{s(i,q)}; W_i(y)\}$. One can trace through the proof of Theorem 4 and verify that $f(z, \underline{a}_1)|T(\beta_1)^{P_1} \cdots T(\beta_r)^{P_r}$ has a Fourier expansion with the same W-system, and so one can conclude that $F \in \underline{F}(n, \chi, W)$. QED

Remark. In the beginning of the proof, we fixed an arbitrary $N^{th}$ root. Thus, there are actually N different $F(z)$ corresponding to $f(z, \underline{a}_1)$, one for each choice of the root. If $F \in \underline{F}(n, \chi)$, changing the root corresponds to changing the character $\chi$.

Remark. If $\left[\pi\beta_1^{P_1} \cdots \beta_r^{P_r}\right]$ is principal, one can verify that the principal component of $T(\beta_1)^{P_1} \cdots T(\beta_r)^{P_r} T(\pi)$ is a sum of matrices from $GL(2, O(K))$, that is, all of whose entries are principal. See Chapter II Section 6 for an example of this. Thus, the hypotheses of Theorem 6 may be stated without using ideal numbers.

## IV.1 Eisenstein Series

Let $n$ be an even integer, and $s$ a complex variable. In the classical case one considers Eisenstein series

$$E(z,s) = \sum_{c,d\in\mathbb{Z}}{}' (cz+d)^{-n}\left(\frac{y}{|cz+d|^2}\right).$$

In analogy to this, we define Eisenstein series on the quaternionic upper half space. Fix an imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d<0$, $d\neq -3,-4$. Let $\underline{I}/\underline{P}$ be the ideal class group with representatives $\underline{a}_1 = (1), \underline{a}_2,\ldots,\underline{a}_h$.

Definition. For $z\in H$ and $s\in\mathbb{C}$ and any ideal classes $[\underline{a}_m],[\underline{a}_p]$, define a quaternionic Eisenstein series of weight $n$ by

$$E(z,s;\underline{a}_m,\underline{a}_p) = \sum_{\substack{\mu\in O[\underline{a}_m]\\ \nu\in O[\underline{a}_p]}}{}' \rho^{-n}(\mu z+\nu)\left(\frac{y}{|\mu z+\nu|^2}\right)^s$$

where $\Sigma'$ means that we eliminate $\mu=\nu=0$.

Consider the action of $\Gamma_{\underline{a}}^{\underline{b}}(\xi)$ on $E(z,s;\underline{a}_m,\underline{a}_p)$. Recall that addition of ideal numbers is only defined within a given $O[\underline{c}]$, hence this action will be defined only if $[\underline{b}] = [\underline{a}_m\underline{a}_p^{-1}\underline{a}]$. Let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_{\underline{a}}^{\underline{b}}(\xi)$. Then one finds that

$$E(z, s; \underline{a}_m, \underline{a}_n) \Big| \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \rho^{-n}(\xi^{-1/2}(\gamma z + \delta))$$

$$\cdot \sum_{\substack{\mu \in O[\underline{a}_m] \\ \nu \in O[\underline{a}_n]}}' \rho^n(\xi^{-1/2}(\gamma z + \delta)) \cdot \rho^{-n}(\xi^{-1/2}[(\mu\alpha + \nu\gamma)z$$

$$+ (\mu\beta + \nu\delta)]) \cdot \left(\frac{y|\xi|}{|(\mu\alpha + \nu\gamma)z + (\mu\beta + \nu\delta)|^2}\right)^s$$

Suppose $\xi = 1$.

<u>Proposition 19</u>

If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^{\underline{a}}_{\underline{a}\,\underline{a}_m \underline{a}_p^{-1}}$, then

$$(O[\underline{a}_m], O[\underline{a}_p]) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (O[\underline{a}_m \underline{a}], O[\underline{a}_p \underline{a}^{-1}]) \quad .$$

<u>Proof.</u> Let $\mu \in O[\underline{a}_m]$, $\nu \in O[\underline{a}_p]$. Then $\alpha\mu + \gamma\nu \in O[\underline{a}_m \underline{a}]$ and $\beta\mu + \delta\nu \in O[\underline{a}_p \underline{a}^{-1}]$. Thus, the right side of the expression above contains the left side. Since the determinant is one, we can invert the matrix, and get the reverse inclusion, hence equality. <u>QED</u>

<u>Proposition 20</u>

For $A \in \Gamma^{\underline{a}}_{\underline{a}\,\underline{a}_m \underline{a}_p^{-1}}$ we have

$$E(z, s; \underline{a}_m, \underline{a}_p) | A = E(z, s; \underline{a}_m \underline{a}, \underline{a}_p \underline{a}^{-1}) .$$

Proof. We have already shown that if $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then

$$E(z, s; \underline{a}_m, \underline{a}_p) \Big| A = \sum_{\substack{\mu \in O[\underline{a}_m] \\ \nu \in O[\underline{a}_n]}}' \rho^{-n}((\mu\alpha + \nu\gamma)z + (\mu\beta + \nu\delta))$$

$$\cdot \left( \frac{y}{|(\mu\alpha + \nu\gamma)z + (\mu\beta + \nu\delta)|^2} \right)^s$$

and now Prop. 19 implies the desired result. QED

Define a vector quaternionic Eisenstein series $F(z, s)$ by

$$F(z, s) = (E(z, s; \underline{a}_m, \underline{a}_1))_{m=1,2,\ldots,h} .$$

As in the last chapter, we will usually not write the indices $m = 1, 2, \ldots, h$. Prop. 20 implies that $F(z, s) | A = F(z, s)$ for $A \in M$.

Let $\chi$ be a character on the ideal class group.

Definition. The vector quaternionic Eisenstein series of weight $n$ and character $\chi$ is defined by

$$F(z, s, \chi) = \frac{1}{h} \sum_{p=1}^{h} \overline{\chi}(\underline{a}_p)\, F(z, s) | C(\underline{a}_p)$$

$$= \left( \frac{1}{h} \sum_{p=1}^{h} \chi(\underline{a}_p)\, E(z, s; \underline{a}_m \cdot \underline{a}_p, \underline{a}_p) \right)_{m=1,2,\ldots,h} .$$

Remark. In terms of the definition of a vector modular form given in Chapter III, $F(z, s, \chi)$ is actually $(n+1)$ different vector forms, one form corresponding to each column of $F(z, s, \chi)$. It will be convenient for us to consider the entire matrix at once, rather than a single column. The reader should be aware, however, that one should actually be viewing each column as a separate modular form.

Remark. One can easily verify that

$$F(z, s, \chi) | C(\underline{a}) = \chi(a)\, F(z, s, \chi)$$

and that

$$F(z, s, \chi) \left| \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \right. = (-1)^{n/2}\, F(z, s, \chi) \ ,$$

so that each column of $F$ belongs to $\underline{F}(n, \chi, \varphi)$ where $\varphi$ is the trivial character if $n \equiv 0(4)$ and the "signum" character if $n \equiv 2(4)$.

Remark. Usually one defines Eisenstein series with the added condition that $(\mu, \nu) = 1$. If we define

$$\widetilde{F}(z, s, \chi, \underline{a}_m) = \frac{1}{h} \sum_{\substack{\mu,\ \nu \in O(Z) \\ [\mu/\nu] = [\underline{a}_m] \\ (\mu, \nu) = 1}} \chi(\nu)\, \rho^{-1}(\mu z + \nu) \left( \frac{y}{|\mu z + \nu|^2} \right)^s ,$$

then one can show that

$$F(z, s, \chi, \underline{a}_m) = \widetilde{F}(z, s, \chi, \underline{a}_m) \cdot L(s, \chi)$$

where $L(s, \chi)$ is an obvious diagonal matrix to be defined below.

## IV.2 Fourier Expansion of Eisenstein Series

To find the Fourier expansion of the classical Eisenstein series, one uses the Poisson summation formula. We need a multi-dimensional version which I adopt from Stark [10].

Consider $\mathbb{R}^q$ with the usual inner product, and take a complete lattice $L$. Let $d\lambda = d\lambda_1\, d\lambda_2 \cdots d\lambda_q$ be the usual Lebesgue measure with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$. Let $d(L)$ be the volume of the fundamental parallelopiped. Define the dual lattice $L'$ by $L' = \{ \lambda' \in \mathbb{R}^q \mid \lambda' \cdot L \subset \mathbb{Z} \}$.

Theorem 7A

Take $f \in C^2(\mathbb{R}^q)$ dying "rapidly" at infinity. Then

$$\sum_{\lambda \in L} f(\lambda) = \frac{1}{d(L)} \sum_{\lambda' \in L'} \int_{\mathbb{R}^q} \cdots \int f(\lambda)\, e^{-2\pi i\, \lambda \cdot \lambda'}\, d\lambda \quad .$$

In our case, we will apply this to the lattice $O[\underline{a}_p]$ in $\mathbb{C}$. Take the usual isomorphism of $\mathbb{C}$ with $\mathbb{R}^2$ via $a + bi \mapsto (a,b)$. Let $tr(x) = x + \overline{x}$ as before. One can verify that if $x = a + bi$ and $y = c + di$ then $(a,b) \cdot (c,d) = tr(x\overline{y})$. One can also show that if $y = cx$ for $c \in \mathbb{C}$, then $dy = |c|^2 dx$.

By definition, the lattice dual to $O(K)$ under the trace is the inverse different $(\delta)^{-1}$, which we can write as $\delta^{-1} \cdot O[\delta]$. Now $O(K) = \overline{O(K)}$ so $\delta^{-1} \cdot O[\delta]$ is also the dual lattice with respect to the inner product of $R^2$. From this it follows that the lattice dual to $O[\underline{a}_p]$ under the trace is $\delta^{-1} \cdot O[\delta \underline{a}_p^{-1}]$ and the lattice dual under the inner product is the complex conjugate of this. Using this, we can rephrase Theorem 7A.

<u>Theorem 7A</u>

For $f \in C^2(C)$ dying "rapidly" at infinity,

$$\sum_{\nu \in O[\underline{a}_p]} f(\nu) = \frac{1}{N(\delta)} \sum_{\lambda \in O[\delta \underline{a}_p^{-1}]} \iint_{\mathbb{C}} f(\nu)\; e\left(-\frac{\nu\lambda}{\delta}\right) d\nu \quad .$$

For the rest of this section, let $\int$ be a short notation for $\iint_{\mathbb{C}}$. Also, $\Sigma'$ means that we eliminate the obvious zero term.

Theorem 8

$$E(z,s;\underline{a}_m,\underline{a}_p) = \sum_{\nu\in O[\underline{a}_p]}{}' \rho^{-n}(\nu)\,\frac{y^s}{|\nu|^{2s}}$$

$$+\frac{1}{N(\delta)}\int\rho^{-n}(u+k)(|u|^2+1)^{-s}\,du \cdot \sum_{\mu\in O[\underline{a}_m]}{}' \rho^{-n}(\mu)\,\frac{N(\mu)}{|\mu|^{2s}}\,y^{2-n-s}$$

$$+\sum_{\nu\in O[\underline{a}_m\underline{a}_p^{-1}]}{}' \frac{1}{N(\delta)}\int\rho^{-n}(u+k)(|u|^2+1)^{-s}\,e\left(-uy\,\frac{\nu}{\delta}\right)du$$

$$\cdot \sum_{\substack{\mu\mid\nu\\ \mu\in O[\underline{a}_m]}} \frac{\rho^{-n}(\mu)\,N(\mu)}{|\mu|^{2s}}\,y^{2-n-s}\;e\left(\frac{\nu x}{\delta}\right).$$

Proof

$$E(z,s;\underline{a}_m,\underline{a}_p) = \sum_{\nu\in O[\underline{a}_p]}{}' \rho^{-n}(\nu)\left(\frac{y}{|\nu|^2}\right)^s$$

$$+\sum_{\mu\in O[\underline{a}_m]}{}' \;\sum_{\nu\in O[\underline{a}_p]} \rho^{-n}(\mu z+\nu)\left(\frac{y}{|\mu z+\nu|^2}\right)^s.$$

We apply Theorem 7 to the inner sum.

$$\sum_{\nu \in O[\underline{a}_p]} \rho^{-n}(\mu z + \nu) \left(\frac{y}{|\mu z + \nu|^2}\right)^s$$

$$= \frac{1}{N(\delta)} \sum_{\lambda \in O[\delta \underline{a}_p^{-1}]} \int \rho^{-n}(\mu z + \nu) \left(\frac{y}{|\mu z + \nu|^2}\right)^s e(-\nu\lambda/\delta)\, d\nu \,.$$

In the integral, substitute $u = (\mu x + \nu)/\mu y$. The integral equals

$$\int \rho^{-n}(\mu y(u+k)) \left(\frac{y}{|\mu y(u+k)|^2}\right)^s \cdot e(-\mu y\; u\,\lambda/\delta)\; e\left(x\,\frac{\mu\lambda}{\delta}\right)$$

$$\cdot\; N(\mu)\, y^2\, du$$

$$= \int \rho^{-n}(u+k)(|u|^2+1)^{-2s}\; e\left(-uy\;\frac{\mu\lambda}{\delta}\right)\, du$$

$$\cdot\; \rho^{-n}(\mu)\; N(\mu)\; |\mu|^{-2s}\; y^{2-n-s} \quad .$$

Putting this into our original expression and separating out the $\lambda = 0$ term, we get

$$E(z, s; \underline{a}_m, \underline{a}_p) = \sum_{\nu \in O[\underline{a}_p]}{}' \rho^{-n}(\nu)\, \frac{y^s}{|\nu|^{2s}}$$

$$+ \sum_{\mu \in O[\underline{a}_m]}{}' \frac{1}{N(\delta)} \int \rho^{-n}(u+k)(|u|^2+1)^{-s}\, du\, \frac{\rho^{-n}(\mu)\, N(\mu)}{|\mu|^{2s}}\, y^{2-n-s}$$

$$+ \sum_{\mu \in O[\underline{a}_m]}{}' \sum_{\lambda \in O[\delta \underline{a}_p^{-1}]} \frac{1}{N(\delta)} \int \rho^{-n}(u+k)(|u|^2+1)^{-s}$$

$$\cdot\ e\left(-uy\,\frac{\mu\lambda}{\delta}\right) du \ \cdot\ \frac{\rho^{-n}(\mu)\,N(\mu)}{|\mu|^{2s}}\, y^{2-n-s}\ e\left(x\,\frac{\mu\lambda}{\delta}\right) .$$

Now let $\nu = \mu\lambda$ in the last summation so that $\nu \in O[\delta \underline{a}_m \underline{a}_p^{-1}]$. After switching summation signs, we get the desired result. QED

Definition. Define an "L-series" $L(s, \chi)$ by

$$L(s, \chi) = \sum_{\nu\, \in O(Z)}{}' \frac{\chi(\nu)\,\rho^{-n}(\nu)}{|\nu|^{2s}} .$$

Define a "divisor function" $D(\nu, s, \chi)$ by

$$D(\nu, s, \chi) = \overline{\chi}(\nu) \sum_{\substack{\mu \mid \nu \\ \mu \in O(Z)}} \frac{\chi(\mu)\,\rho^{-n}(\mu)\,N(\mu)}{|\mu|^{2s}} .$$

Set

$$W(\nu y, s) = \frac{1}{N(\delta)} \int \rho^{-n}(u+k)(|u|^2+1)^{-s}\ e\left(-u\,\frac{\nu y}{\delta}\right) du .$$

Corollary to Theorem 8

Let $F(z, s, \chi) = (F(z, s, \chi, \underline{a}_m))$ be the vector Eisenstein series defined before. Then we have a Fourier expansion

$$F(z, s, \chi, \underline{a}_m) = \frac{1}{h} L(s, \chi)\, y^s$$

$$+ \frac{1}{h}\,\overline{\chi}(\underline{a}_m)\, W(0, s)\ L(s-1, \chi)\, y^{2-n-s}$$

$$+\frac{1}{h}\sum_{\nu\in O[\delta\underline{a}_m]}{}' \chi(\delta)\, W(\nu y, s)\, D(\nu, s, \chi)\, y^{2-n-s}\, e\left(\frac{\nu x}{\delta}\right).$$

Proof. Put the Fourier expansion of Theorem 8 into

$$\frac{1}{h}\sum_{p=1}^{h} \chi(\underline{a}_p)\, E(z, s; \underline{a}_m, \underline{a}_p)\ . \qquad \text{QED}$$

## IV.3 Action of the Hecke Operators

Lemma 5

Let $\pi$ be prime. Then

$$E(z, s; \underline{a}_m, \underline{a}_p)\Big| T_{[\underline{a}_m\underline{a}_p^{-1}]}(\pi)$$

$$= E(z, s; \underline{a}_m, \underline{a}_p\pi)\ |\pi|^{s+n/2}\ \rho^n(\pi^{1/2})\ N(\pi)^{-1}$$

$$+ E(z, s; \underline{a}_m\pi^{-1}, \underline{a}_p)\cdot |\pi|^{-s+n/2}\ \rho^{-n}(\pi^{1/2})\ .$$

Proof

$$E(z, s; \underline{a}_m, \underline{a}_p)\Big| T_{[\underline{a}_m\underline{a}_p^{-1}]}(\pi)$$

$$= E(z, s; \underline{a}_m, \underline{a}_p)\left|\ \left\{ C^{\pi^{-1}}_{\pi^{-1}\underline{a}_m\underline{a}_p^{-1}}\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix} + \sum_{\substack{\sigma \bmod \pi\\ \sigma\in O[\underline{a}_m^{-1}\underline{a}_p\pi]}}\begin{pmatrix}1 & \sigma\\ 0 & \pi\end{pmatrix}\right\}\right.$$

$$\cdot\ |\pi|^{n/2}\ N(\pi)^{-1}\ .$$

Prop. 20 along with the definition of the slash operator shows that this becomes

$$= \sum'_{\substack{\mu \in O[\underline{a}_m \pi^{-1}] \\ \nu \in O[\underline{a}_p \pi]}} \rho^{-n}(\pi^{-1/2}(\pi\mu z + \nu)) \left(\frac{y|\pi|}{|\pi\mu z + \nu|^2}\right)^s$$

$$\cdot\ |\pi|^{n/2}\, N(\pi)^{-1}$$

$$+ \sum_{\substack{\sigma \bmod \pi \\ \sigma \in O[\underline{a}_m^{-1} \underline{a}_p \pi]}} \sum'_{\substack{\mu \in O[\underline{a}_m] \\ \nu \in O[\underline{a}_p]}} \rho^{-n}(\pi^{-1/2}[\mu z + (\mu\sigma + \nu\pi)])$$

$$\cdot \left(\frac{y|\pi|}{|\mu z + (\mu\sigma + \nu\pi)|^2}\right)^s |\pi|^{n/2}\, N(\pi)^{-1} \quad .$$

Break the second sum into a sum over $\pi | \mu$ and a sum of $\pi \nmid \mu$. For a fixed $\pi \nmid \mu$, with all $\sigma \bmod \pi$ and all $\nu \in O[\underline{a}_p]$, we find that $\mu\sigma + \nu\pi$ gives every element of $O[\underline{a}_p \pi]$. Thus we replace $\mu\sigma + \nu\pi$ by the variable $\nu$ to get

$$= \sum'_{\substack{\mu \in O[\underline{a}_m \pi^{-1}] \\ \nu \in O[\underline{a}_p \pi]}} \rho^{-n}(\pi\mu z + \nu) \left(\frac{y}{|\pi\mu z + \nu|^2}\right)^s |\pi|^{s+n/2}\, N(\pi)^{-1}$$

$$\cdot\ \rho^{n}(\pi^{1/2})$$

$$+ \sum_{\substack{\pi \nmid \mu \\ \mu \in O[\underline{a}_m] \\ \nu \in O[\underline{a}_p \pi]}} \rho^{-n}(\mu z + \nu) \left(\frac{y}{|\mu z + \nu|^2}\right)^s |\pi|^{s+n/2} \rho^n(\pi^{1/2}) N(\pi)^{-1}$$

$$+ \sum_{\substack{\sigma \bmod \pi \\ \sigma \in O[\underline{a}_m^{-1} \underline{a}_p \pi]}} \sum_{\substack{\pi \mid \mu \\ \mu \in O[\underline{a}_m \pi^{-1}] \\ \nu \in O[\underline{a}_p]}}' \rho^{-n}(\mu z + (\mu\sigma + \nu\pi))$$

$$\cdot \left(\frac{y}{|\mu z + (\mu\sigma + \nu\pi)|^2}\right)^s \cdot |\pi|^{s+n/2} \rho^n(\pi^{1/2}) N(\pi)^{-1} \quad .$$

The first two sums combine to become

$$E(z, s; \underline{a}_m, \underline{a}_p \pi) \, |\pi|^{s+n/2} \rho^n(\pi^{1/2}) N(\pi)^{-1} \quad .$$

In the final sum, $\pi \mid \mu$ so replace $\mu$ by $\pi\mu$ where now $\mu \in O[\underline{a}_m \pi^{-1}]$. This last term now becomes

$$\sum_{\sigma \bmod \pi} \sum_{\substack{\mu \in O[\underline{a}_m \pi^{-1}] \\ \nu \in O[\underline{a}_p]}}' \rho^{-n}(\pi[\mu z + (\mu\sigma + \nu)])$$

$$\cdot \left(\frac{y}{|\pi|^2 \, |\mu z + (\mu\sigma + \nu)|^2}\right)^s |\pi|^{s+n/2} \rho^n(\pi^{1/2}) N(\pi)^{-1} \quad .$$

For a fixed $\sigma$, $(\mu, \nu) \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} = (\mu, \mu\sigma + \nu)$, so by Prop. 19,

$\mu\sigma + \nu$, and noting that the sum over $\sigma \bmod \pi$ is trivial, we see that the final sum becomes

$$= N(\pi) \sum_{\substack{\mu \in O[\underline{a}_m \pi^{-1}] \\ \nu \in O[\underline{a}_p]}}{}' \rho^{-n}(\mu z + \nu) \left(\frac{y}{|\mu z + \nu|^2}\right)^s |\pi|^{-s+n/2}$$

$$\cdot\ \rho^{-n}(\pi^{1/2})\, N(\pi)^{-1}$$

$$= E(z, s; \underline{a}_m \pi^{-1}, \underline{a}_p)\, |\pi|^{-s+n/2}\ \rho^{-n}(\pi^{1/2})$$

which gives the desired result. QED

Theorem 9

For any $\mu \in O(Z)$,

$$F(z, s, \chi)|T(\mu) = F(z, s, \chi)\ D(\mu, s, \chi)\, |\mu|^{s+n/2}\ \rho^{n}(\mu^{1/2})\, N(\mu)^{-1}\,.$$

Proof. First we will show that it is true for $\mu = \pi$ prime, and then for prime powers, and finally for all $\mu$.

Let $\pi$ be prime. Then

$$F(z, s, \chi, \underline{a}_m)|T_{[\underline{a}_m]}(\pi) = \frac{1}{h}\sum_{p=1}^{h} \chi(\underline{a}_p)\ E(z, s; \underline{a}_m \underline{a}_p, \underline{a}_p)|T_{[\underline{a}_m]}(\pi)$$

$$= \frac{1}{h}\sum_{p=1}^{h} \overline{\chi}(\pi)\ \chi(\pi \underline{a}_p)\ E(z, s; \underline{a}_m \pi^{-1} \underline{a}_p \pi, \underline{a}_p \pi)$$

$$\cdot\ |\pi|^{s+n/2}\ \rho^{n}(\pi^{1/2})\ N(\pi)^{-1}$$

$$+\frac{1}{h}\sum_{p=1}^{h}\chi(\underline{a}_p)\ E(z,s;\underline{a}_m\pi^{-1}\underline{a}_p,\underline{a}_p)$$

$$\cdot\ |\pi|^{-s+n/2}\ \rho^{-n}(\pi^{1/2})$$

$$= F(z,s,\underline{a}_m\pi^{-1})\cdot D(\pi,s,\chi)\,|\pi|^{s+n/2}\,\rho^{n}(\pi^{1/2})\,N(\pi)^{-1}$$

where we have used Lemma 5 and skipped much of the tedious calculation.

Now we will use Theorem 2′ to find the action of the rest of Hecke operators. For prime powers, we use induction. If we set $\lambda(\pi^q)$ to be the matrix eigenvalue for $T(\pi^q)$, then we want to show that

$$\lambda(\pi^q) = D(\pi^q,s,\chi)\,|\pi^q|^{s+n/2}\,\rho^{n}(\pi^{q/2})\,N(\pi^q)^{-1} \quad .$$

We have shown the result for $q = 1$. Assume it is true for $q - 1$, $q \geq 2$. Then Theorem 2′ implies that

$$\lambda(\pi)\,\lambda(\pi^{q-1}) = \lambda(\pi^q) + N(\pi^{-1})\,\overline{\chi}(\pi)\,|\pi|^n\,\lambda(\pi^{q-2})$$

so using our inductive hypothesis and some algebraic manipulations, we find that

$$\lambda(\pi^q)\,|\pi^q|^{-s-n/2}\,\rho^{-n}(\pi^{q/2})\,N(\pi^q)$$

$$= D(\pi, s, \chi)\,D(\pi^{q-1}, s, \chi) - \overline{\chi}(\pi)\,|\pi|^{-2s}\,\rho^{-n}(\pi)\,N(\pi)$$

$$\cdot\, D(\pi^{q-2}, s, \chi)\ .$$

We need to show that the right hand side is $D(\pi^q, s, \chi)$. Tedious but trivial algebraic manipulation paralleling that in the proof of Theorem 2 establishes this fact, which then gives the desired result for prime powers.

Consider now the general case. If $(\mu, \nu) = 1$, then $\lambda(\mu)\,\lambda(\nu) = \lambda(\mu\nu)$ so we need to show that $D(\mu, s, \chi)\,D(\nu, s, \chi) = D(\mu\nu, s, \chi)$. This is easily verified. Now we can consider any $\mu, \nu \in O(Z)$ by considering each prime factor separately, and then combining them using our last statement to show that

$$\lambda(\mu\nu) = D(\mu\nu, s, \chi)\,|\mu\nu|^{s+n/2}\,\rho^n((\mu\nu)^{1/2})\,N(\mu\nu)^{-1}\ . \qquad \underline{\text{QED}}$$

Remark. Once again recall that in terms of the last chapter, $F(z, s, \chi)$ is actually $(n+1)$ different eigenforms, which is why the "eigenvalue" for $F(z, s, \chi)$ is actually an $(n+1) \times (n+1)$ diagonal matrix. The entries of this matrix are easily calculated via Prop. 9.3.

Remark. The "eigenvalue" $\lambda(\mu)$ may be rewritten as

$$\left\{ \sum_{\substack{\theta \mid \mu \\ \theta \in O(Z)}} \chi(\theta)\, \rho^{n}(\theta)\, N(\theta)^{-1}\, |\theta|^{2s} \right\} |\mu|^{-s+n/2}\, \rho^{-n}(\mu^{1/2})$$

The term in brackets looks remarkably similar to the divisor function, namely $\sigma_{n-1}(q) = \sum_{t \mid q} t^{n-1}$, which appears in the eigenvalues of the classical Eisenstein series.

## IV.4 Dirichlet Series Associated to Eisenstein Series

In this section, I will not repeat the proofs of Chapter III Section 7; instead, I will specify what the results of that section yield when applied to the quaternionic Eisenstein series.

We begin with the Mellin transform. Unfortunately, I have used the variable $s$ in the Eisenstein series, so I will take the Mellin transform with respect to the complex variable $r$. The proof of Theorem 5 and Corollary 8 show that

$$\int_0^\infty y^r \left[ \frac{1}{h} \sideset{}{'}\sum_{\nu \in O[\delta \underline{a}_m]} \chi(\delta)\, W(\nu y, s)\, D(\nu, s, \chi)\, y^{2-n-s} \right] \frac{dy}{y}$$

$$= \rho^n(k)\, \overline{\chi}(\underline{a}_m) \int_0^\infty y^{n-r} \left[ \frac{1}{h} \sideset{}{'}\sum_{\nu \in O[\delta \underline{a}_m^{-1}]} \chi(\delta) \right.$$

$$\cdot\ W(\nu y, s)\ D(\nu, s, \chi)\, y^{2-s-n}\Big]\ \frac{dy}{y}\ .$$

Recall that

$$W(\nu y, s) = \frac{1}{N(\delta)} \iint_{\mathbb{C}} \rho^{n}\left(\left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)\overline{u} - k\right)$$

$$\cdot\ (|u|^{2}+1)^{-s-n}\ e\left(-y\left|\frac{\nu}{\delta}\right| u\right)\ du\ .$$

We substitute $y\left|\frac{\nu}{\delta}\right|^{-1}$ for $y$ so that the Mellin transforms become

$$\sum_{\nu\in O[\delta \underline{a}_m]}{}' \frac{1}{hN(\delta)} \int_{0}^{\infty} y^{r+2-s-n} \iint_{\mathbb{C}} \rho^{n}\left(\left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)\overline{u} - k\right)$$

$$\cdot\ (|u|^{2}+1)^{-s-n}\ e(-yu)\, du\ \cdot\ \left|\frac{\nu}{\delta}\right|^{s+n-r-2}\ D(\nu, s, \chi)\ \frac{dy}{y}$$

$$= \rho^{n}(k)\ \overline{\chi}(\underline{a}_m) \sum_{\nu\in O[\delta \underline{a}_m^{-1}]}{}' \frac{1}{hN(\delta)} \int_{0}^{\infty} y^{2-r-s}$$

$$\cdot \iint_{\mathbb{C}} \rho^{n}\left(\left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)\overline{u} - k\right)\ (|u|^{2}+1)^{-s-n}\ e(-yu)\, du$$

$$\cdot\ \left|\frac{\nu}{\delta}\right|^{2-r-s}\ D(\nu, s, \chi)\ \frac{dy}{y}\ .$$

Now will consider each entry of this matrix equation separately. One can verify as in Prop. 9.1 that $\rho^n\left(\left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)\bar{u}-k\right)$ is a matrix whose $ij^{th}$ entry is of the form

$$\left\{\left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)\right\}^{n+2-i-j} P_{ij}(u,\bar{u})$$

where $P_{ij}(u,\bar{u})$ is a polynomial with rational integer coefficients. One can use Prop. 9.2 to show that the $j^{th}$ diagonal entry of $D(\nu, s, \chi)$ is

$$(D(\nu,s,\chi))_j = \overline{\chi}(\nu) \sum_{\substack{\mu \mid \nu \\ \mu \in O(Z)}} \chi(\mu)\, \bar{\mu}^{n+1-j}\, \mu^{j-1}\, |\mu|^{-2s-2n}\, N(\mu) \ .$$

For any $i,j = 1,2,\ldots,n+1$, define a Dirichlet series

$$D_{ij}(r,s,\underline{a}_m,\chi) = \sum_{\nu \in O[\delta \underline{a}_m]}{}' \left(\frac{\nu}{|\nu|}\right)^{n+2-i-j} |\nu|^{s+n-2-r}$$

$$\cdot\ (D(\nu,s,\chi))_j \ .$$

Let $V_{ij}(r,s)$ be defined by

$$V_{ij}(r,s) = \left(\frac{\delta}{|\delta|}\right)^{i+j-2-n} \frac{|\delta|^{r+2-s-n}}{hN(\delta)} \int_0^\infty y^{r+2-s-n}$$

$$\cdot \iint_{\mathbb{C}} P_{ij}(u,\bar{u})(|u|^2+1)^{-s-n}\, e(-yu)\, du\, \frac{dy}{y} \ .$$

With these definitions, we get a matrix functional equation

$$(D_{ij}(r, s, \underline{a}_m, \chi)\, V_{ij}(r, s)) = \rho^n(k)\, \overline{\chi}(\underline{a}_m)$$

$$\cdot\, (D_{ij}(n-r, s, \underline{a}_m^{-1}, \chi)\, V_{ij}(n-r, s)) \quad .$$

We can also define a Dirichlet series using the Hecke eigenvalues. Theorem 9 shows that the eigenvalue for $T(\mu)$ has $j^{th}$ diagonal entry

$$\lambda_j(\mu) = (D(\mu, s, \chi))_j\, |\mu|^{s+n/2}\, N(\mu)^{-1}\, \mu^{\frac{1}{2}(n+1-j)}\, \mu^{\frac{1}{2}(j-1)} \quad .$$

Let $\varphi$ be a character on U, with $\varphi$ the trivial character when $n \equiv 0 \bmod 4$, and $\varphi$ the non-trivial "signum" character when $n \equiv 2 \bmod 4$. Take $\psi$ to be any character of the ideal class group. Set

$$\psi_i(\mu) = \psi(\mu)\, \varphi(\mu) \left(\frac{\mu}{|\mu|}\right)^{n/2+1-i} \quad .$$

Then

$$\Delta_j(r, s, \chi, \psi_i) = \frac{1}{2} \sum_{\mu \in O(Z)} \frac{c_{ij}(\mu)\, \psi(\mu)}{|\mu|^r}$$

$$= \frac{\psi(\delta)}{2} \sum_{m=1}^{h} \psi(\underline{a}_m)\, D_{ij}(r, s, \underline{a}_m, \chi)$$

$$= \prod_{(\pi)\,\text{prime}} \left( 1 - \frac{c_{ij}(\pi)\ \psi(\pi)}{|\pi|^2} + \frac{N(\pi)^{-1}\ \overline{\chi}(\pi)\ \psi(\pi^2)}{|\pi|^{2r-n}} \cdot \left(\frac{\pi}{|\pi|}\right)^{n+2(1-i)} \right)^{-1}$$

where

$$c_{ij}(\mu) = \left(\frac{\mu}{|\mu|}\right)^{n+2-i-j} |\mu|^{s+n-2}\ (D(\mu, s, \chi))_j$$

for any $\mu \in O(Z)$. The functional equation becomes

$$(\overline{\psi}(\delta)\ \Delta_j(r, s, \chi, \psi_i)\ V_{ij}(r, s)) = \rho^n(k)$$

$$\cdot\ ((\overline{\chi}\psi)(\delta)\ \Delta_j(n-r, s, \chi, (\chi\overline{\psi})_i)\ V_{ij}(n-r, s)) \quad .$$

IV.5 Explicit Evaluation to Get Bessel Functions

We are now interested in explicitly calculating the matrix of functions $W(\nu y, s)$. The interested reader is urged to see Patterson-Goldfeld [8] for a very slick method of evaluation. This section uses a more direct approach which relies only on a special functions table, such as Gradshteyn-Ryzhik [3].

First, we perform a series of transformations to simplify the integrals. Recall that

$$W(\nu y, s) = \frac{1}{N(\delta)} \iint_{\mathbb{C}} \rho^{-n}(u+k)(|u|^2+1)^{-s} e^{-2\pi i \operatorname{tr}\left(uy \frac{\nu}{\delta}\right)} du \ .$$

By Prop. 8, $\rho^{-n}(u+k) = \rho^{n}(\bar{u}-k)(|u|^2+1)^{-n}$. Let $u$ be replaced by $\left|\frac{\nu}{\delta}\right|\left(\frac{\nu}{\delta}\right)^{-1} u$. This transformation leaves $|u|^2$ and $du$ unchanged. Replace $4\pi y\left|\frac{\nu}{\delta}\right|$ by a new variable $w$. The integral now becomes

$$\frac{1}{N(\delta)} \iint_{\mathbb{C}} \rho^{n}\left(\left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)\bar{u}-k\right)(|u|^2+1)^{-s-n} e^{-\frac{i}{2}\operatorname{tr}(wu)} du \ .$$

By Prop. 9.1, $\rho^{n}\left(\left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)\bar{u}-k\right)$ is a matrix whose entries are polynomials in $\bar{u}$ and $u$ with coefficients involving power of $\left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)$. In any given case, one can easily calculate these polynomials. We can now evaluate any entry of $W(\nu y, s)$ provided that we can evaluate integrals of the form

$$\iint_{\mathbb{C}} u^{p_1}\bar{u}^{p_2}(|u|^2+1)^{-s-n} e^{-\frac{i}{2}\operatorname{tr}(wu)} du$$

for integral powers $p_1$ and $p_2$.

Let $u = a + ib$ so $\operatorname{tr} u = 2a$ and $du = da\,db$. Then

$u^{p_1} \bar{u}^{p_2}$ is a polynomial in $a$ and $b$ with rational integer coefficients. Again we conclude that one can evaluate the above integral provided that one can evaluate

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^p b^q (a^2 + b^2 + 1)^{-s-n} e^{-iwa} \, db \, da$$

for rational integers $p$ and $q$. Set $c = \dfrac{b}{\sqrt{a^2+1}}$. Then this integral becomes

$$\int_{-\infty}^{\infty} a^p (a^2+1)^{\frac{1}{2}(q+1)-s-n} e^{-iwa} \, da \cdot \int_{-\infty}^{\infty} \frac{c^q \, dc}{(c^2+1)^{s+n}} .$$

Consider the second integral. If $q$ is odd, then $\dfrac{c^q}{(c^2+1)^{s+n}}$ is an odd function, so the integral evaluates to zero. Thus we may let $q = 2q$ for some integral $q$. Now we find that $2\displaystyle\int_0^{\infty} \frac{c^{2q} \, dc}{(c^2+1)^{s+n}}$ is the beta function, which evaluates to

$$\frac{\Gamma\left(q+\frac{1}{2}\right) \Gamma\left(s+n-q-\frac{1}{2}\right)}{\Gamma(s+n)} .$$

Now we consider the remaining integral,

$$\int_{-\infty}^{\infty} a^p e^{-iwa}(a^2+1)^{q+\frac{1}{2}-s-n} da .$$

Replace $e^{iwa}$ by $\cos wa + i \sin wa$. Suppose that $p$ is even, say $p = 2p$. Then $a^{2p} \sin wa$ is an odd function so its contribution to the integral is zero. Consider

$$\int_{-\infty}^{\infty} \frac{a^{2p} \cos wa\, da}{(a^2+1)^{s+n-q-1/2}} .$$

We replace $a^{2p}$ by $[(a^2+1) - 1]^p$ so that the integral becomes a finite sum of integrals

$$\int_{-\infty}^{\infty} \frac{\cos wa\, da}{(a^2+1)^{s+n-v+1/2}}$$

for $v$ rational integral. Fortunately, Gradshteyn-Ryzhik [3] contain this integral in their section on Bessel functions. Specifically,

$$\int_{-\infty}^{\infty} \frac{\cos wa\, da}{(a^2+1)^{s+n-v+1/2}} = \frac{2\, w^{s+n-v}\, \Gamma\left(\frac{1}{2}\right)}{2^{s+n-v}\, \Gamma\left(s+n-v+\frac{1}{2}\right)} K_{s+n-v}(w)$$

where $K$ is the K-Bessel function.

We are left with the case of $p$ odd, say $p = 2t + 1$. Then $a^p \cos wa$ is an odd function, so its contribution to the integral is zero. We again let $a^p = a^{2t+1} = a[(a^2+1) - 1]^t$ so we need only evaluate terms of the form

$$i \int_{-\infty}^{\infty} \frac{a \sin wa \, da}{(a^2+1)^{s+n-v+1/2}}$$

for $v$ rational integral. Applying the chain rule and using the identity quoted above, one can show that

$$i \int_{-\infty}^{\infty} \frac{a \sin wa \, da}{(a^2+1)^{s+n-v+1/2}} = \frac{2\,i\, w^{s+n-v}\, \Gamma\left(\frac{1}{2}\right)}{2^{s+n-v}\, \Gamma\left(s+n-v+\frac{1}{2}\right)} K_{s+n-v-1}(w) \;.$$

Now we put together all of our results. We find that the original matrix $W(\nu y, s)$ has entries which are finite sums of terms of the form

$$\frac{\pi\, w^{s+n-t}}{2^{s+n-t}\, \Gamma(s+n)} K_{s+n-q}(w) \;.$$

Here $q$ and $t$ are rational integers. The coefficients of the sums are rational function of $s$,

$$\left|\frac{\nu}{\delta}\right|^{-1} \left(\frac{\nu}{\delta}\right), \quad \text{and} \quad i = \sqrt{-1} \;.$$

IV.6 Example of $W(\nu y, s)$ when $n = 2$

Set $n=2$ and let $v = \left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)$ for some $\nu \in O(Z)$. We will explicitly find $W(\nu y, s)$. First I list several useful equations.

$$\rho^2(v\bar{u} - k) = \begin{pmatrix} v^2\bar{u}^2 & -2v\bar{u} & 1 \\ vu & |u|^2 - 1 & -\bar{v}u \\ 1 & 2\bar{v}u & \bar{v}^2u^2 \end{pmatrix} .$$

For any $p$ such that the following converge,

$$\int_{-\infty}^{\infty} \frac{\cos wa\, da}{(a^2+1)^{p+1/2}} = \frac{2w^p\, \Gamma\left(\frac{1}{2}\right)}{2^p\, \Gamma\left(p+\frac{1}{2}\right)} K_p(w)$$

$$\int_{-\infty}^{\infty} \frac{a \sin wa\, da}{(a^2+1)^{p+1/2}} = \frac{2w^p\, \Gamma\left(\frac{1}{2}\right)}{2^p\, \Gamma\left(p+\frac{1}{2}\right)} K_{p-1}(w)$$

$$\int_{-\infty}^{\infty} \frac{a^2 \cos wa\, da}{(a^2+1)^{p+1/2}} = \frac{2w^{p-1}\, \Gamma\left(\frac{1}{2}\right)}{2^{p-1}\, \Gamma\left(p-\frac{1}{2}\right)} K_{p-1}(w) - \frac{2w^p\, \Gamma\left(\frac{1}{2}\right)}{2^p\, \Gamma\left(p+\frac{1}{2}\right)} K_p(w)$$

$$\int_{-\infty}^{\infty} \frac{dc}{(c^2+1)^{s+n}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s+n-\frac{1}{2}\right)}{\Gamma(s+n)}$$

$$\int_{-\infty}^{\infty} \frac{c^2\, dc}{(c^2+1)^{s+n}} = \frac{\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(s+n-\frac{3}{2}\right)}{\Gamma(s+n)} .$$

If we let $u = a + ib$, then $u^2 = a^2 + 2iab - b^2$. Also, the

K-Bessel functions satisfy the recursion relation

$$wK_{p-1}(w) - wK_{p+1}(w) = -2p\, K_p(w)\,.$$

Using these equations, one can verify that

$$\iint_{\mathbb{C}} u^2(|u|^2+1)^{-s-n}\, e^{-\frac{i}{2}\operatorname{tr}(wu)}\, du = -\frac{2\pi w^{s+n-1}}{2^{s+n-1}\,\Gamma(s+n)}\, K_{s+n-3}(w)\,.$$

$$\iint_{\mathbb{C}} u(|u|^2+1)^{-s-n}\, e^{-\frac{i}{2}\operatorname{tr}(wu)}\, du = -i\,\frac{2\pi w^{s+n-1}}{2^{2+n-1}\,\Gamma(s+n)}\, K_{s+n-2}(w)\,.$$

$$\iint_{\mathbb{C}} (|u|^2+1)^{-s-n}\, e^{-\frac{i}{2}\operatorname{tr}(wu)}\, du = \frac{2\pi w^{s+n-1}}{2^{s+n-1}\,\Gamma(s+n)}\, K_{s+n-1}(w)\,.$$

Now let $w = 4\pi y \left|\frac{\nu}{\delta}\right|$ and $v = \left|\frac{\nu}{\delta}\right|^{-1}\left(\frac{\nu}{\delta}\right)$. Then

$$W(\nu y, s) = \frac{2\pi w^{s+n-1}}{2^{s+n-1}\, N(\delta)\, \Gamma(s+n)}\,.$$

$$\cdot\begin{pmatrix} -v^2 K_{s+n-3}(w) & 2iv K_{s+n-2}(w) & K_{s+n-1}(w) \\ -ivK_{s+n-2}(w) & -K_{s+n-3}(w) + \frac{2}{w} K_{s+n-2}(w) - K_{s+n-1}(w) & i\bar{v}K_{s+n-2}(w) \\ K_{s+n-1}(w) & -2i\bar{v} K_{s+n-2}(w) & -\bar{v}^2 K_{s+n-3}(w) \end{pmatrix}$$

## IV. 7 Differential Equations for Weight One

We have only dealt with even weights so that we could avoid multiplier systems. Nevertheless, we will now consider a weight one quaternionic-valued Eisenstein series, defined by

$$E_1(z, s; \underline{a}_m, \underline{a}_p, M) = \sum_{\substack{\gamma \in O[\underline{a}_m] \\ \delta \in O[\underline{a}_p]}}' M(\gamma, \delta)(\gamma z + \delta)^{-1} \left(\frac{y}{|\gamma z + \delta|^2}\right)^s$$

where $M$ is a suitable multiplier system. I will not need $M$ explicitly. The interested reader may see Goldfeld [2]. We follow a paper of Maass [6] to establish the following theorem.

Theorem 10

Let $z \in H$ with $z = u + iv + ky$, with $u, v, y \in \mathbb{R}$. Then

$$\left\{-y^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial y^2}\right) + ky\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right) + s^2 - s\right\}$$

$$\cdot\ [(\gamma z + \delta)^{-1} y^s |\gamma z + \delta|^{-2s}] = 0$$

for any $\gamma, \delta \in \mathbb{C}$. $E_1(z, s; \underline{a}_m, \underline{a}_p, M)$ satisfies the same differential equation for any $m, p, M$.

Proof. The theorem follows from straightforward but laborious calculation. One can show the following equations:

Let $A = |\gamma z + \delta|^2$.

$$\frac{\partial}{\partial u} A^{-s-1} = -2(s+1)(|\gamma|^2 u + \operatorname{Re} \overline{\gamma}\delta)\ A^{-s-2}$$

$$\frac{\partial}{\partial v} A^{-s-1} = -2(s+1)(|\gamma|^2 v + \operatorname{Im} \overline{\gamma}\delta)\ A^{-s-2}$$

$$\frac{\partial}{\partial y} A^{-s-1} = -2y\,|\gamma|^2 (s+1)\ A^{-s-2}$$

$$\frac{\partial^2}{\partial u^2} A^{-s-1} = -2(s+1)\,|\gamma|^2 A^{-s-2} + 4(s+1)(s+2)$$

$$\cdot\ (|\gamma|^2 u + \operatorname{Re} \overline{\gamma}\delta)^2\ A^{-s-3}$$

$$\frac{\partial^2}{\partial v^2} A^{-s-1} = -2(s+1)\,|\gamma|^2\ A^{-s-2} + 4(s+1)(s+2)$$

$$\cdot\ (|\gamma|^2 v + \operatorname{Im} \overline{\gamma}\delta)^2\ A^{-s-3}$$

$$\frac{\partial^2}{\partial y^2} A^{-s-1} = -2(s+1)\,|\gamma|^2 A^{-s-2} + 4(s+1)(s+2)(|\gamma|^2 y)^2\ A^{-s-3}$$

$$\frac{\partial}{\partial u}\ \overline{(\gamma z + \delta)} = \overline{\gamma}$$

$$\frac{\partial}{\partial v}\ \overline{(\gamma z + \delta)} = -i\,\overline{\gamma}$$

$$\frac{\partial}{\partial y}\ \overline{(\gamma z + \delta)} = -k\,\overline{\gamma}\,.$$

One can show that

$$\overline{\gamma}(|\gamma|^2 u + \mathrm{Re}\,\overline{\gamma}\delta) - i\overline{\gamma}(|\gamma|^2 v + \mathrm{Im}\,\overline{\gamma}\delta) - ky|\gamma|^2\,\overline{\gamma}$$

$$= |\gamma|^2\,\overline{(\gamma z + \delta)} \qquad \text{and also that}$$

$$(|\gamma|^2 u + \mathrm{Re}\,\overline{\gamma}\delta)^2 + (|\gamma|^2 v + \mathrm{Im}\,\overline{\gamma}\delta)^2 + (|\gamma|^2 y)^2 = |\gamma|^2 A \ .$$

Now $(\gamma z + \delta)^{-1} y^s |\gamma z + \delta|^{-2s} = \overline{(\gamma z + \delta)}\, y^s A^{-s-1}$ . Thus

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial y^2}\right) ((\overline{\gamma z + \delta})\, y^s A^{-s-1})$$

$$= 2\overline{\gamma}\, y^s \frac{\partial}{\partial u} A^{-s-1} + \overline{(\gamma z + \delta)}\, y^s \frac{\partial^2}{\partial u^2} A^{-s-1}$$

$$+ 2(-i\overline{\gamma})\, y^s \frac{\partial}{\partial v} A^{-s-1} + \overline{(\gamma z + \delta)}\, y^s \frac{\partial^2}{\partial v^2} A^{-s-1}$$

$$+ 2(-k\overline{\gamma})(sy^{s-1}) A^{-s-1} + 2(-k\overline{\gamma})\, y^s \frac{\partial}{\partial y} A^{-s-1}$$

$$+ \overline{(\gamma z + \delta)}\, s(s-1) y^{s-2} A^{-s-1} + 2\overline{(\gamma z + \delta)}\, sy^{s-1} \frac{\partial}{\partial y} A^{-s-1}$$

$$+ \overline{(\gamma z + \delta)}\, y^s \frac{\partial^2}{\partial y^2} A^{-s-1} \ .$$

One can now verify that when we put in the appropriate results from above, we get

$$= -4[\overline{\gamma}(|\gamma|^2 u + \mathrm{Re}\ \overline{\gamma}\delta) - i\overline{\gamma}(|\gamma|^2 v + \mathrm{Im}\ \overline{\gamma}\delta) - k\overline{\gamma}y|\gamma|^2]$$

$$\cdot (s+1)y^s A^{-s-2} + 2(-k\overline{\gamma})\ sy^{s-1} A^{-s-1}$$

$$+ \overline{(\gamma z + \delta)}y^s[-6(s+1)|\gamma|^2 A^{-s-2}]$$

$$+ \overline{(\gamma z + \delta)}|\gamma|^2 y^s \cdot 4(s+1)(s+2)[(|\gamma|^2 u + \mathrm{Re}\ \overline{\gamma}\delta)^2 + (|\gamma|^2 v + \mathrm{Im}\ \overline{\gamma}\delta)^2$$

$$+ (|\gamma|^2 y)^2] \cdot A^{-s-3}$$

$$+ \overline{(\gamma z + \delta)}\ s(s-1)\ y^{s-2} A^{-s-1}$$

$$+ 2\overline{(\gamma z + \delta)}\ sy^{s-1}[-2y|\gamma|^2 (s+1) A^{-s-2}]$$

Now one uses the results mentioned above along with the following identity to simplify everything:

$$-4(s+1) - 6(s+1) + 4(s+1)(s+2) - 4s(s+1)$$

$$= -2(s+1)$$

The above expression now becomes

$$= -2\overline{(\gamma z + \delta)}|\gamma|^2(s+1)y^s A^{-s-2} - 2k\overline{\gamma}s\,y^{s-1} A^{-s-1}$$

$$+ \overline{(\gamma z + \delta)}\ s(s-1)\ y^{s-2} A^{-s-1} \quad .$$

Similarly, one can verify that

$$\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right)\left[\overline{(\gamma z+\delta)}\,y^s A^{-s-1}\right] = 2\overline{\gamma}\,y^s A^{-s-1}$$

$$- 2\overline{(\gamma z+\delta)}\,y^s(s+1)\left(|\gamma|^2 u + \mathrm{Re}\,\overline{\gamma}\delta\right) A^{-s-2}$$

$$- 2i\overline{(\gamma z+\delta)}\,y^s(s+1)\left(|\gamma|^2 v + \mathrm{Im}\,\overline{\gamma}\delta\right) A^{-s-2}$$

One can show that $k|\gamma|^2 y = \overline{\gamma}(\gamma z+\delta)$

$$- \left(|\gamma|^2 u + \mathrm{Re}\,\overline{\gamma}\delta\right) - i\left(|\gamma|^2 v + \mathrm{Im}\,\overline{\gamma}\delta\right) .$$

Using this along with $A = (\gamma z+\delta)\overline{(\gamma z+\delta)}$ shows that the equation becomes

$$\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right)\left[\overline{(\gamma z+\delta)}\,y^s A^{-s-1}\right] = -2\overline{\gamma}s\,y^s A^{-s-1}$$

$$+ 2k\,|\gamma|^2\,\overline{(\gamma z+\delta)}(s+1)y^{s+1} A^{-s-2} .$$

Now one can put all the terms together and can verify that

$$\left\{-y^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial y^2}\right) + ky\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right) + s^2 - s\right\}\left[\overline{(\gamma z+\delta)}\,y^s A^{-s-1}\right]$$

$$= 0 .$$

Since $E_1(z, s; \underline{a}_m, \underline{a}_p, M)$ is a linear sum of such terms, it follows that it satisfies the same differential equation. QED

In analogy with our previous results, one might suspect that $E_1$ has a Fourier expansion consisting of terms like

$$B\left(4\pi\left|\frac{\nu}{\delta}\right|y\right)\ e\left(\frac{\nu x}{\delta}\right)$$

for some function B. If we put this term into our differential equation, and if we let $w = 4\pi\left|\frac{\nu}{\delta}\right|y$ as before, then

$$-w^2\frac{\partial^2}{\partial w^2}B(w) + \left(w^2 + kw\left|\frac{\nu}{\delta}\right|\left(\frac{\nu}{\delta}\right)^{-1} i + s^2 - s\right)B(w) = 0 \ .$$

One might hope that B would be the sum of Bessel functions, say $B(w) = B_1(w) + kB_2(w)$. Unfortunately, a minus sign which arises in commuting quaternions prevents this from happening. This is consistent with the results of the last sections, where we could have shown that the matrix of Bessel functions arising from

$$\frac{1}{N(\delta)}\iint_{\mathbb{C}} \rho(v\bar{u} - k)\, e^{-\frac{i}{2}\operatorname{tr}(wu)}\, du$$

is (up to scalar factors) the matrix

$$\begin{pmatrix} i\bar{v}\ K_{s+n-2}(w) & K_{s+n-1}(w) \\ -K_{s+n-1}(w) & iv\ K_{s+n-2}(w) \end{pmatrix} .$$

This matrix does not represent a quaternion, precisely because of a missing minus sign. We will see this phenomenon again in the next section.

Another approach is to recognize that since the quaternion $k$ is the special basis element, we should look for a Fourier expansion with terms

$$B\left(8\pi\left|\frac{\nu}{\delta}\right| y\right) e^{2\pi k \operatorname{tr}\left(\frac{\nu x}{\delta}\right)} .$$

Note the change of 4 to 8 in the argument of B. We now get an equation for B

$$\frac{\partial^2}{\partial w^2} B(w) + \left(-\frac{1}{4} + \frac{\frac{1}{2}v}{w} + \frac{\frac{1}{4} - \left(s^2 - s + \frac{1}{4}\right)}{w^2}\right) B(w) = 0$$

where

$$w = 8\pi\left|\frac{\nu}{\delta}\right| y \quad \text{and} \quad v = \left|\frac{\nu}{\delta}\right|^{-1} \left(\frac{\nu}{\delta}\right) .$$

According to Gradshteyn-Ryzhik [3], this is the equation for the Whittaker functions $W_{\lambda,\mu}(w)$ with $\lambda = \pm\frac{v}{2}$ and $\mu = (s^2 - s + 1/4)^{1/2}$. It is well-known that the Whittaker functions appear in the Fourier coefficients of the classical Eisenstein series, so our result is not surprising.

## IV. 8 Differential Equations for Higher Weights

For weights greater than one, we have defined Eisenstein series with a matrix representation, hence we expect to have a matrix of differential equations. We have shown that up to constants a general term looks like $y^{2-n-s}\,W(\nu y, s)\,e\left(\frac{\nu x}{\delta}\right)$. One can fairly easily find an equation for the $y^{2-n-s}\,e\left(\frac{\nu x}{\delta}\right)$ terms, and so we are most interested in finding the differential equations for $W(\nu y, s)$. As before, we can reduce any entry of the matrix $W(\nu y, s)$ to a sum of integrals

$$\iint_{\mathbb{C}} u^p\,\bar{u}^q(|u|^2+1)^{-s-n}\,e^{-\frac{i}{2}\operatorname{tr}(wu)}\,du$$

where $w = 4\pi y\left|\frac{\nu}{\delta}\right|$.

Now we use a slightly modified idea from Patterson-Goldfeld [8]. Suppose we consider $w$ to be a complex variable, $w = w_1 + iw_2$, so that the above integral is the limit as $w_2 \to 0$. Now $\operatorname{tr} wu = wu + \bar{w}\bar{u}$ so

$$\frac{\partial}{\partial w}\operatorname{tr}(wu) = u \qquad \text{and} \qquad \frac{\partial}{\partial \bar{w}}\operatorname{tr} wu = \bar{u}$$

where

$$\frac{\partial}{\partial w} = \frac{\partial}{\partial w_1} - i\frac{\partial}{\partial w_2} \qquad \text{and} \qquad \frac{\partial}{\partial \bar{w}} = \frac{\partial}{\partial w_1} + i\frac{\partial}{\partial w_2} .$$

Then

$$\iint_{\mathbb{C}} \iota^{p} u^{q} (|\overline{u}|^2+1)^{-s-n} e^{-\frac{i}{2}\operatorname{tr}(wu)} du$$

$$= (2i)^{p+q} \left(\frac{\partial}{\partial w}\right)^{p}\left(\frac{\partial}{\partial \overline{w}}\right)^{q} \iint_{\mathbb{C}} (|u|^2+1)^{-s-n} e^{-\frac{i}{2}\operatorname{tr}(wu)} du \quad .$$

The interesting term is the $i^{p+q}$. If we would have gotten $i^{p}(-i)^{q}$, then one could have shown that the differential equations would be given by

$$\rho^{n}\left(2i\frac{\partial}{\partial \overline{w}} - k\right) \iint_{\mathbb{C}} (|u|^2+1)^{-s-n} e^{-\frac{i}{2}\operatorname{tr}(wu)} du \quad .$$

One may of course "refuse" or "forget" to conjugate the 2i factor when finding the entries of $\rho^{n}\left(2i\frac{\partial}{\partial \overline{w}} - k\right)$ and then one does get the appropriate differential equations with factors of $i^{p+q}$ rather than $i^{p}(-i)^{q}$.

Note that

$$\iint_{\mathbb{C}} (|u|^2 + 1)^{-s-n} e^{-\frac{i}{2}\operatorname{tr} wu} du$$

is essentially a Bessel function, so differentiating with respect to w still gives a Bessel function. Thus, we have verified again that every entry of $W(\nu y, s)$ is a sum of Bessel functions.

An amusing observation arises out of this approach. One can show that the invariant Laplacian for H, the quaternionic upper half space, is

$$y^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial y^2}\right)+y\frac{\partial}{\partial y} \quad \text{where} \quad z = u+iv+ky \in H .$$

The question is why the linear $y\frac{\partial}{\partial y}$ term appears.

Let B be a function of a positive real variable. Suppose that we consider $y$ to be a complex variable with $y = y_1 + iy_2$. Set $|y| = (y_1^2 + y_2^2)^{1/2}$ as usual, and look at $\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right) B(|y|)$.

We get that

$$\frac{\partial^2}{\partial y_1^2}B(|y|) + \frac{\partial^2}{\partial y_2^2}B(|y|)$$

$$= \frac{\partial}{\partial y_1}\left(\frac{\partial |y|}{\partial y_1}\frac{\partial}{\partial |y|}B(|y|)\right) + \frac{\partial}{\partial y_2}\left(\frac{\partial |y|}{\partial y_2}\frac{\partial}{\partial |y|}B(|y|)\right) .$$

Both terms are essentially the same, so consider the second.

$$\frac{\partial}{\partial y_2}\left(\frac{y_2}{|y|}\frac{\partial}{\partial |y|}B(|y|)\right) = \frac{\partial}{\partial y_2}\left(\frac{y_2}{|y|}\right)\frac{\partial}{\partial |y|}B(|y|)$$

$$+\frac{y_2}{|y|}\frac{\partial |y|}{\partial y_2}\frac{\partial^2}{\partial |y|^2}B(|y|)$$

$$= \frac{y_1^2}{|y|^3} \frac{\partial}{\partial |y|} B(|y|) + \frac{y_2^2}{|y|^2} \frac{\partial^2}{\partial |y|^2} B(|y|) \quad .$$

Thus,

$$\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right) B(|y|) = \frac{1}{|y|} \frac{\partial}{\partial |y|} B(|y|) + \frac{\partial^2}{\partial |y|^2} B(|y|) \quad .$$

This calculation may indicate why the linear term appears in the invariant Laplacian of H.

## IV. 9 Example: Eisenstein Series over $K = \mathbb{Q} = (\sqrt{-23})$

Recall the notation of Chapter II Section 6. We let $O_0$ be the ideal numbers $O[(1)]$, and $O_1$ be the ideal numbers $O[\underline{p}_2]$, and $O_2$ be the ideal numbers $O[\underline{p}_2^2]$. In particular, $\mu \in O_0$ says that $\mu \in O(K)$, $\mu \in O_1$ says that $\pi_2^2 \mu \in \underline{p}_2^2$, and $\mu \in O_2$ says that $\pi_2 \mu \in \underline{p}_2$. Define

$$E(z, s; a, b) = \sum_{\substack{\mu \in O_a \\ \nu \in O_b}}' \rho^{-n}(\mu z + \nu) \left(\frac{y}{|\mu z + \nu|^2}\right)^s$$

for $a, b = 0, 1, 2$.

Let $a' \equiv (3 - a) \bmod 3$, $b' \equiv (3 - b) \bmod 3$, with $0 \le a', b' \le 2$. Then $\mu z = (\mu \pi_2^{a'})(\pi_2^{-a'} z)$ and $\nu = (\nu \pi_2^{b'})(\pi_2^{-b'})$ and if $\mu \in O_a$ and $\nu \in O_b$ then $\mu \pi_2^{a'} \in \underline{p}_2^{a'}$ and $\nu \pi_2^{b'} \in \underline{p}_2^{b'}$.

Thus,

$$E(z, s; a, b) = \sum_{\substack{c \in \underline{p}_2^{a'} \\ d \in \underline{p}_2^{b'}}}' \rho^{-n}(c(\pi_2^{b'-a'} z) + d)$$

$$\cdot \left(\frac{y}{|c(\pi_2^{b'-a'} z) + d|^2}\right)^s \rho^n(\pi_2^{b'}) \; |\pi_2^{b'}|^{2s}$$

where now the variables $c$ and $d$ are in $O(K)$.

Let $\chi$ be a cubic character on $\{0, 1, 2\}$. Define $F(z, s, \chi) = (F(z, s, \chi, a))_{a = 0, 1, 2}$ by

$$F(z, s, \chi, a) = \frac{1}{3} \sum_{b=0}^{2} \chi(b) \; F(z, s, a+b, b) \; .$$

In particular,

$$F(z, s, \chi, 0) = \frac{1}{3} \sum_{c, d \in O(K)} \rho^{-n}(cz + d) \left(\frac{y}{|cz + d|^2}\right)^s$$

$$+ \frac{1}{3} \sum_{c, d \in \underline{p}_2^2} \chi(1) \; \rho^{-n}(cz + d) \left(\frac{y}{|cz + d|^2}\right)^s \rho^n(\pi_2^2) \; |\pi_2^2|^{2s}$$

$$+ \frac{1}{3} \sum_{c, d \in \underline{p}_2} \chi(2) \; \rho^{-n}(cz + d) \left(\frac{y}{|cz + d|^2}\right)^s \rho^n(\pi_2) \; |\pi_2|^{2s} \; .$$

One might think that the principal component $F(z, s, \chi, 0)$ depends on the choice of ideal number $\pi_2$, which is only determined up to a sixth root of unity. One sees, however, that $|\pi_2|^{2s}$ is independent of the root of unity, and since $n$ is even, $\rho^n(\pm\zeta_3\pi_2) = (\pm\zeta_3)^n \rho^n(\pi_2)$ with $(\pm\zeta_3)^n = \zeta_3^n$ being a cubic root of unity. Thus we can define a new character $\chi'$ by $\chi'(a) = \chi(a)\zeta_3^{na'}$. Thus, changing $\pi_2$ by a root of unity is equivalent to changing the character of $F(z, s, \chi, 0)$. In other words, the family of functions $\{F(z, s, \chi, 0) | \chi$ a cubic character$\}$ is defined independently of the definition of the ideal numbers $Z$. By Theorem 6 (the principal theorem), each $F(z, s, \chi, 0)$ essentially determines some vector form $F(z, s, \chi')$ where the ambiguity in the cubic root can be resolved by changing $\chi'$. We conclude that the family of vector modular forms $\{F(z, s, \chi) | \chi$ cubic character$\}$ is defined independently of the particular definition of $Z$.

We now consider Dirichlet series associated with $K = \mathbb{Q}(\sqrt{-23})$. Note that $\delta = \sqrt{-23}$ is principal. For convenience, assume $n \equiv 0(4)$ so that the unit character is trivial. Let $\chi$ and $\psi$ be characters of the ideal class group. Then one can perform tandard manipulations of Euler products to show that

$$\Delta_j(r, s, \chi, \psi_i) = L(r+s, \psi, j-i)\, L(r-s-n+2, \overline{\chi}\psi, n+2-i-j)$$

where we define the "L-series" by

$$L(r, \chi, j) = \frac{1}{2} \sum_{\mu \in O(Z)} |\mu|^{-r} \chi(\mu) \left(\frac{\mu}{|\mu|}\right)^j$$

where $r \in \mathbb{C}$, $\chi$ is any character, and $j$ is a rational integer. Just as we did above for the Eisenstein series, one can write these L-series in terms of the ideal number $\pi_2$ and principal elements of $O(K)$. For instance,

$$L(r+s, \psi, j-i) = \frac{1}{2} \sum_{\nu \in O(K)} |\nu|^{-r-s} \left(\frac{\nu}{|\nu|}\right)^{j-i}$$

$$+\frac{1}{2} |\pi_2|^{2(r+s+j-i)} \pi_2^{2(i-j)} \psi(\pi_2) \sum_{\nu \in \underline{p}_2^2} |\nu|^{-r-s} \left(\frac{\nu}{|\nu|}\right)^{j-i}$$

$$+\frac{1}{2} |\pi_2|^{r+s+j-i} \pi_2^{i-j} \psi(\pi_2^2) \sum_{\nu \in \underline{p}_2} |\nu|^{-r-s} \left(\frac{\nu}{|\nu|}\right)^{j-i} .$$

Once again, as with the Eisenstein series, we may consider the effect of replacing $\pi_2$ by $\pm\zeta_3 \pi_2$, with $\zeta_3^3 = 1$. If we redefine the "positive" ideal numbers in terms of the new $\pi_2$, then $L(r+s, \psi, j-i)$ changes by $(\pm 1)^{i-j} \zeta_3^{i-j}$. We can redefine the cubic character $\psi$ as before to remove the $\zeta_3^{i-j}$ factor. One can show that for $a = 0, 1, 2$,

$$\sum_{\nu \in \underline{p}_2^a} |\nu|^{-r-s} \left(\frac{\nu}{|\nu|}\right)^{j-i} = 0 \text{ whenever } (-1)^{j-i} = -1 \ .$$

Thus, one concludes that the family of L-series $\{L(r+s, \psi, j-i) \mid \psi$ any cubic character$\}$ is defined independently of the definition of the ideal number $\pi_2$. From this it follows that the family of Dirichlet series $\{\Delta_j(r, s, \chi, \psi_i) \mid \chi, \psi$ cubic characters$\}$ is also defined independently of $\pi_2$.

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