Smallest examples of strings of consecutive happy numbers

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Abstract

A happy number N is defined by the condition $S^n(N) = 1$ for some number n of iterations of the function S, where $S(N)$ is the sum of the squares of the digits of N. Up to 10^{20}, the longest known string of consecutive happy numbers was length five. We find the smallest string of consecutive happy numbers of length 6, 7, 8, ..., 13. For instance, the smallest string of six consecutive happy numbers begins with $N = 78999999999999959999999996$. We also find the smallest sequence of 3-consecutive cubic happy numbers of lengths 4, 5, 6, 7, 8, and 9.

1 Introduction

Many recreational math problems deal with various combinations of digits. Let $N = \sum_{j=0}^{n} a_j 10^j$ with $0 \le a_j \le 9$ be the decimal expansion of N, define $S(N) = \sum_{j=0}^{n} a_j^2$, and define S^k to be the k-fold iterate of S. Honsberger [3] (see also Beardon [1] and Grundman and Teeple [4]) gives a simple proof that, for every positive integer N there is a positive integer k such that either $S^k(N) = 1$, a fixed point, or $S^k(N) = 4$ which is part of the 8-cycle $\{16, 37, 58, 89, 145, 42, 20, 4\}$.

We say a number is happy if the iterates converge to the fixed point 1. The first happy number bigger than 1 is 7, and indeed, it seems that about 1 out of 7 numbers is happy [7]. The smallest pair of consecutive numbers is 31 and 32, and the smallest with 3, 4 or 5 consecutive happy numbers were known; for example, 44488, 44489, 44490, 44491, 44492 is the smallest string of five consecutive happy numbers. Guy [7] notes that Jud McCranie calculated that up to 10^{20} the longest consecutive string is only of length five. Guy asks how many consecutive happy numbers one can have. El-Sedy and Siksek [2] published the first proof that there can be arbitrarily long strings of consecutive happy numbers (although H. Lenstra knew a proof earlier; when Teeple was working on her undergraduate honors thesis, she wrote Lenstra who responded in June 1997 that he had a proof in his files. Teeple generalized his proof in her undergraduate thesis [10].) See also Grundman and Teeple [5],

[6], and Pan [8]. Their techniques do not come remotely close to finding the least examples, however, and we propose to find the smallest instance of six or more consecutive happy numbers.

J. A. Littlewood said "A technique is a trick used more than once." In their paper on happy numbers, El-Sedy and Siksek [2] end their paper by using a trick to calculate a huge number $l = \sum_{r=1}^{233192} 9 \cdot 10^{4+r} + 20958$ with certain properties that are critical to their proof. We can transform their trick into a technique that could be used to calculate the minimal value for a number l with their desired properties, namely, $l = 4699999990999999999969$.

We will illustrate this technique by find the smallest N beginning a sequence of 6 through 13 consecutive happy numbers. We use a dot as the concatenation operator, and list the number of nine digits in parentheses. For example, $N = 58.(11 \text{ nines}).6.(144 \text{ nines}).5$ means

$$N = 58 \cdot 10^{157} + 10^{146}(10^{11} - 1) + 6 \cdot 10^{145} + 10(10^{144} - 1) + 5$$

that is, a 159 digit number given by the digits 58 followed by eleven digits 9, then the digit 6, then one hundred forty-four digits 9, and ending with the digit 5. In this table, n is the length of the sequence of consecutive happy numbers, *digits* is the number of digits in each member of the sequence, and N is the first number of the sequence.

n	*digits*	N
2	2	31
3	4	1880
4	4	7839
5	5	44488
6	25	7899999999999959999999996
7	25	7899999999999959999999996
8	159	58.(11 nines).6.(144 nines).5
9	215	26.(137 nines).7.(74 nines).5
10	651	38.(560 nines).0.(87 nines).5
11	1571	27.(280 nines).0.(1287 nines).4
12	158162	388.(158021 nines).8.(136 nines).4
13	603699	288.(218491 nines).3.(385203 nines).3

2 Methodology

Proposition 1 $N_0 = 7899999999999959999999996$ *is the smallest number that begins a sequence of six consecutive happy numbers.*

It is easy to verify that each of these is a happy number:
7899999999999959999999996
7899999999999959999999997
7899999999999959999999998
7899999999999959999999999
7899999999999960000000000
7899999999999960000000001
7899999999999960000000002

Note that these are 25 digit numbers, so if there were a smaller N beginning a sequence of six consecutive happy numbers, it must have $S(N) < 2025 = 25 \cdot 9^2$. El-Sedy and Siksek [2] (their Lemma 2.3) note the following:

Lemma 1 *If N_1 and N_2 are natural numbers with $N_2 < 10^k$ then $S(N_1 * 10^k + N_2) = S(N_1) + S(N_2)$.*

Let $N \leq 7899999999999959999999996$ begin a string of six consecutive happy numbers. Let $N = N_1.d_0$ where d_0 is the final digit of N, and N_1 is formed from the rest of the digits. We organize our proof by looking at the various possibilities for d_0.

Let $M_1 = S(N_1)$. Then $S(N) = M_1 + S(d_0)$ and as we just noted, we may assume $M_1 < 2025$.

Lemma 2 *Let $N \leq 7899999999999959999999996$ and $N = N_1.d_0$. When $d_0 = 0$, one can have at most three successive happy numbers, N, $N+1$, $N+2$.*

Proof: Suppose the last digit of N is $d_0 = 0$. Note that $S(N) = S(N_1) = M_1$, $S(N+1) = S(N_1)+1^2 = M_1+1^2$, $S(N+2) = S(N_1)+2^2 = M_1+2^2$, and $S(N+3) = S(N_1)+3^2 = M_1+3^2$. We simply calculate that M_1, $M_1 + 1^2$, $M_1 + 2^2$ and $M_1 + 3^2$ are not simultaneously happy for any $M_1 < 2025$ (for the relevant Maple programs, see [9]). For use in a later lemma, note that we can to extend this search to all $M_1 < 2025 + 17$.

Lemma 3 *Let $N \leq 7899999999999959999999996$ and $N = N_1.d_0$. Suppose $d_0 = 1$, 2, 3, 4 or 5. Then there are at most four consecutive happy numbers beginning at N.*

Proof: We check if $M_1+d_0^2$, $M_1+(d_0+1)^2$, $M_1+(d_0+2)^2$, $M_1+(d_0+3)^2$ and $M_1+(d_0+4)^2$ are simultaneously happy for any value of $M_1 \leq 2025$; our calculations show there are at most four consecutive happy numbers for N with $d_0 = 1$, 2, 3, 4 or 5.

Lemma 4 *Let $N \leq 7899999999999959999999996$ and $N = N_1.d_0$. Suppose $d_0 = 8$ or 9. Then there are at most five consecutive happy numbers beginning at N.*

Proof: Suppose N has final digit $d_0 = 8$. Then $N+2$ would have final digit 0, and $N+2$ begins a sequence of four consecutive happy numbers. Similarly, if $d_0 = 9$ then $N + 1$ has final digit 0 and begins a sequence of five consecutive happy numbers. Either case contradicts Lemma 2. This proves the lemma.

Thus, the final digit of N must be 6 or 7, and there must a carry at $N + 4$ or $N + 3$, respectively.

Suppose N ends in the digit $d_0 = 6$. We check for four consecutive happy numbers, that is, we check if $M_1 = 6^2$, $M_1 + 7^2$, $M_1 + 8^2$, and $M_1 + 9^2$ are happy for any $M_1 < 2025$. The only case is $M_1 = 1839$.

Suppose N ends in the digit $d_0 = 7$. We check for three consecutive happy numbers, that is, we check if $M_1 + 7^2$, $M_1 + 8^2$, and $M_1 + 9^2$ are happy for any $M_1 < 2025$. We find three cases: $M_1 = 568$, 574, and of course 1839.

We now show that $M_1 = 568$ cannot yield three happy numbers after the carry. Note in this case that N must have last digit 7 so $N + 3 = (N_1 + 1).0$. Set $N_1 = N_2.d.9\ldots9$

where digit $d \le 8$, there are exactly k digits of 9 ending N_1, and by convention $N_2 = 0$ if N_2 is empty and $N_2 = 0$ and $d = 0$ if both are empty. Then $S(N_1) = S(N_2) + d^2 + 9^2k$ so $S(N_2) = S(N_1) - d^2 - 9^2k$, and $N_1 + 1 = N_2.(d+1).0\ldots0$ with k zeros in the $0\ldots0$, so $S(N_1+1) = S(N_2)+(d+1)^2$ and $S(N_1+1) = S(N_1)+(d+1)^2-d^2-9^2k = S(N_1)+(2d+1)-9^2k$. Since $M_1 = S(N_1) = S(N_2) + d^2 + 9^2k = 568$, $9^2k \le 568$ so $k \le 7$. We now look for values (k, d), with $0 \le k \le 7$ and $0 \le d \le 8$, for which these three are happy:

$S(N+3) = M_1 + 2d + 1 - 81k + 0^2$
$S(N+4) = M_1 + 2d + 1 - 81k + 1^2$
$S(N+5) = M_1 + 2d + 1 - 81k + 2^2$

A simple computer search shows this never happens. Similarly, when $M_1 = 574$ there are not three consecutive happy numbers "after the carry".

Now consider $M_1 = 1839$. In this case N has last digit $d_0 = 6$. We search for values (k, d), with $k \le 22$ and $d \le 8$, for which these two are happy:

$S(N+4) = M_1 + 2d + 1 - 81k + 0^2$
$S(N+5) = M_1 + 2d + 1 - 81k + 1^2$

There are five possibilities, each giving a string of six or more consecutive happy numbers:

k	d	$S(N_2)$	least N_2 for $S(N_2)$	N_1
1	6	1722	27799999999999999999999	2779999999999999999999969
5	6	1398	2779999999999999999	2779999999999999999699999
9	5	1085	78999999999999	789999999999995999999999
15	6	588	277999999	2779999996999999999999999
19	0	300	57899	5789909999999999999999999

In the next section we will show how we calculated the fourth column. We see that the smallest N_1 corresponds to $k = 9$ and $d = 5$. Thus, we have proven that the smallest N beginning a sequence of six happy numbers is 789999999999995999999996. Unexpectedly, one finds that this extends to seven consecutive happy numbers. This completes the proof of our proposition.

Similar arguments prove that the values of N given in the table in Section 1 are indeed the smallest beginning a sequence of 8, 9, 10, 11, 12, and 13 consecutive happy numbers. Maple worksheets on the author's homepage [9] contain the calculations. The optimal example for 14 consecutive happy numbers seems to be the 34567901197532 digit number 7.(2098518518492 nines).8.(32469382679037 nines).3 which is clearly too large to be calculated by our techniques.

3 Minimal N with $S(N) = n$

Given a positive integer n, we want to determine the least N such that $S(N) = n$. Define $L(n) = \min\{N | S(N) = n\}$. Clearly, if $L(n_1) = L(n_2)$ then $S(L(n_1)) = n_1 = S(L(n_2)) = n_2$, hence if $n_1 \ne n_2$ then $L(n_1) \ne L(n_2)$. Also, $L(n)$ will not have any zero digits and also must have digits in increasing order. We now show how to calculate $L(n)$.

Lemma 5 *We can calculate $L(n_0)$ for all $n_0 < 486$. For $n \ge 486$, let $q = \lfloor \frac{n}{81} \rfloor$ and $n_0 = n - 81(q-5)$. Then $L(n) = L(n_0) \cdot 10^{q-5} + (10^{q-5} - 1)$.*

Proof: We first show that, except for the digit 9, no other digit can be repeated very often in an $L(n)$. Note that $S(1111) = S(2)$, $S(2222) = S(4)$, $S(333) = S(115)$, $S(4444) = S(8)$, $S(55) = S(17)$, $S(6666) = S(488)$, $S(7777) = S(2888)$ and $S(88888888) = S(15999999)$. Thus, if $L(n)$ does not contain the digit 9, then it cannot exceed

$$1112223344456667778888888.$$

Note that $S(1112223344456667778888888) = 809$. We now find all values of $L(n)$ for $n \leq 809$ inductively. Let $R(N)$ be the function that sorts the digits of N from lowest to highest, e.g., $R(526845) = 245568$. The basic idea is to split off the last digit to get a smaller value, so we look at $L(n - d^2) \cdot 10 + d$ for each digit d. If we know $L(k)$ for all $k < n$ then $L(n) = \min_{d=0,1,\ldots,9}\{R(L(n - d^2) \cdot 10 + d)\}$. We thus can find all $L(n)$ for $n \leq 809$.

Our results show that 448 is the largest n for which $L(n)$ has no digit 9. If $n > 448$ then $L(n)$ ends with at elast one digit of nine. Thus, $L(n) = L(n-9^2)\cdot 10+9$, and if $n-81 > 448$ then ?$L(n) = (L(n-9^2-9^2)\cdot 10+9)\cdot 10+9 = L(n-2\cdot 9^2)\cdot 10^2+(10^2-1)$, and by induction $L(n) = L(n_0) \cdot 10^{q-5} + (10^{q-5} - 1)$ where $n_0 = n - (q - 5)9^2 \leq 486$. Thus, $L(n)$ is simply $q - 5$ digits of 9 concatenated to the end of $L(n_0)$. This ends our proof.

It is possible that $n_2 > n_1$ and $L(n_2) < L(n_1)$, for instance, $L(243) = 999 < L(7) = 1112$. This extreme difference of $243-7 = 236$ does not happen again, but note that $L(162+9^2m) < L(54 + 9^2m)$ for all nonnegative integers m, so $L(n_2) < L(n_1)$ with $n_2 - n_1 = 108$ does occur infinitely often.

In the next section, we will extend our methods to cubic happy numbers. Define $L_3(n) = \min\{N|S_{3,10}(N) = n\}$ where $S_{3,10}(\sum_{i=0}^{n} a_i 10^i) = \sum_{i=0}^{n} a_i^3$. We can show that the cubic case analog to the quadratic happy number $N = 1112223344456667778888888$ above is

$$N = 1111111222222333344444555566677777788888888$$

which has $S(N) = 8297$. As before, one inductively calculates each $L_3(n)$ for $n \leq 8279$. Arguments analogous to those in Lemma 5 show the following:

Lemma 6 *We can calculate $L_3(n_0)$ for all $n_0 \leq 5832$. If $n > 4609$ then $L_3(n)$ must contain the digit 9. Let $q = \lfloor n/9^3 \rfloor$ and $n_0 = n - (q - 7)9^3 < 8 \cdot 9^3 = 5832$. Then $L_3(n) = L_3(n_0) \cdot 10^{q-7} + (10^{q-7} - 1)$.*

Although we will not use this, here is a table of the largest n for which $L(n)$ does not contain certain digits, and the corresponding table for $L_3(n)$.

largest allowable digit in $L(n)$	n_{max}	$L(n_{max})$
1	3	111
2	12	222
3	23	1233
4	48	444
5	48	444
6	112	2666
7	151	2777
8	448	8888888

largest allowable digit in $L_3(n)$	n_{max}	$L_3(n_{max})$
1	7	1111111
2	50	11222222
3	124	223333
4	329	1244444
5	572	245555
6	932	555566
7	2183	5777777
8	4609	1888888888

4 Sequences of Generalized Happy Numbers

Grundman and Teeple [4] define the generalized e-power b-happy numbers in terms of the generalized digit power function

$$S_{e,b}(\sum_{i=0}^{n} a_i b^i) = \sum_{i=0}^{n} a_i^e$$

where $0 \le a_i < b$ are the base b digits. If for some m we have $S_{e,b}^m(N) = 1$ then we say N is an e-power b-happy number. The classic happy numbers have $e = 2$ and $b = 10$, and the well-studied cubic happy numbers have $e = 3$ and $b = 10$. In general, one cannot have successive generalized happy numbers, and one can at best expect them to form an arithmetic sequence with arithmetic difference $d = \gcd(e, b-1)$. In particular, any cubic happy number must be congruent to 1 mod 3. So Grundman and Teeple define a d-consecutive sequence as an arithmetic sequence with common difference d; they prove one can find arbitrarily long such sequences for many choices of $\{e, b\}$, in particular, for $e = 3$ and $b = 10$ with $d = 3$.

Our methods can be extended to find the least 3-consecutive sequence of cubic happy numbers. A naive search shows that the smallest 3-consecutive sequence of two cubic happy numbers is $\{1198, 1201\}$, and that the smallest of length three is $\{169957, 169960, 169963\}$. Here is a table of results:

n	digits	N
2	4	1198
3	6	169957
4	16	1555599999999916
5	29	35588899999799999999999999989
6	101	28888.(21 nines).1.(72 nines).89
7	234	3577.(228 nines).45
8	242	1126.(229 nines).1.(6 nines).89
9	276	12777.(151 nines).5.(117 nines).86

We will illustrate our method by discussing how to obtain the smallest 3-consecutive sequence of six cubic happy numbers. Since our N has 101 digits, we do not need to check any N with $S(N) > 101 \cdot 9^3 = 73629$. Whereas in the classic happy number case we split off the final digit, in the cubic happy number case it is more convenient to split off the final two digits. Let $N = N_1.d_1.d_0$ where $0 \le d_0, d_1 \le 9$ are the last two digits. For convenience let $d = d_1.d_0$. We first compute all sequences of five cubic happy numbers

$\{M_1+S(d), M_1+S(d+3), M_1+S(d+6), M_1+S(d+9), M_1+S(d+12)\}$ with $d<85$ and $M_1 = S(N_1) \leq 73629$. There are none.

Similar calculations show that only two sequences $\{M_1+S(d), M_1+S(d+3), M_1+S(d+6), M_1+S(d+9)\}$ of length four with $d = 88, 89, 90$ have $M_1 = 16736$, $d = 89$ or $M_1 = 69854$, $d = 89$. As above, we decompose $N_1 = P_2.d_2.9\cdots9$ where d_2 is a digit not equal 9, and there are exactly k digits of 9 ending N_1. Thus, $N+3 = N_1.92$, $N+6 = N_1.95$, $N+9 = N_1.98$, while $N+12 = P_2.(d_2+1).0\cdots0.01$ and $N+15 = P_2.(d_2+1).0\cdots0.04$ where the $0\cdots0$ is a string of exactly k zeros. Since $S(N) = M_1+8^3+9^3 = S(P_2)+d_2^3+k\cdot9^3+8^3+9^3$, one can see that $S(N+12) = S(P_2)+(d_2+1)^3+1^3 = (M_1-d_2^3-k\cdot9^3)+(d_2+1)^3+1^3$ and similarly $S(N+15) = M_1-d_2^3-k\cdot9^3+(d_2+1)^3+4^3$. When $M_1 = 16736$, there is a k for which $S(N+12)$ is happy (this in fact gives the least example of a 3-consecutive sequence of five cubic happy numbers) but it does not extend to $S(N+15)$. Fortunately, when $M_1 = 69854$, both $k=1, d_2=1$ and $k=72, d_2=1$ yield cubic happy numbers $S(N+12)$ and $S(N+15)$. Using Lemma 6, we can calculate that the smaller N comes from $k=72$ and $d_2=1$, resulting in $N = 28888\cdot10^{96}+(10^{21}-1)\cdot10^{75}+1\cdot10^{74}+(10^{72}-1)\cdot10^2+89$.

To complete the analysis, we must show that there is not a sequence that "splits three before the carry and three after", that is, $N = N_1.d$ with $d = 91$, 92, or 93. The only $M_1 < 73629$ and $d = 91$ or 92 or 93 that give three cubic happy numbers $\{M_1+S(d), M_1+S(d+3), M_1+S(d+6)\}$ are $M_1 = 45001$, $d = 93$, or $M_1 = 54019$, $d = 93$. Decomposing into the P_2 and the string of nine digits as above, we can analyze these cases and verify they cannot be extended to $N+12$ and $N+15$. Thus, our previous value of N is indeed the lowest that can generate a 3-consecutive sequence of six cubic happy numbers.

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