Strings of "Consecutive" Cubic Happy Numbers
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We will investigate consecutive cubic happy numbers. But they can only be "consecutive" in an arithmetic progression with common difference 3. We will abuse language by saying "five in a row" when we really mean a 3-consecutive sequence of length five.

Note that Dr. Grundman uses S_{3,10} but we will simply use S in the explanation, which in our program is the procedure 'onestep'. We will proceed as we did with the classic happy numbers but this time we split off the last two digits rather than just one digit. This makes the loops ten times longer but keeps the things simple. The a variable below corresponds to the

lowest 2 in a row are 1198 and 1201
lowest 3 in a row are 169957, 169960, 169963
We are seeking lowest with four in a row.
Claim: N = 66888899999956
For five in a row, it is
$N := 355888999997999999999999999989$

Also note we will use a dot as the digit concatenation operator. Thus, 111112999994 could be written 11111.2.9999.94 if we wish.

First we define our procedures.

```
> restart;
> f := n → n^3;  #in case we ever want to investigate the cube of the digits, etc.
```

$$f := n \rightarrow n^3 \quad (1)$$

```
> bs := 10;  #this is the base, in case we ever want to investigate binary or ternary or other base.
```

$$bs := 10 \quad (2)$$

```
> onestep := proc(n1)
        #this is what Dr. Grundman calls S_{3,10}(n1) and what we will simply call S below.
    local ans, n, d;
    n := n1;
    ans := 0;
    while n > 0 do
    d := n mod bs;
    ans := ans + f(d);
    n := (n-d)/bs;
    end do;
    ans;
    end;
```

```
onestep := proc(n1)                                                    (3)
    local ans, n, d;
```

```
    n := n1;
    ans := 0;
    while 0 < n do d := mod(n, bs); ans := ans + f(d); n := (n - d) / bs end do;
    ans
end proc
```

```
> happy := proc(n)
      # returns -1 if not happy, and returns the number of steps to reach 1 if it is happy
  local m, j, height;
  m := n;
  height := -1;
  for j from 1 to 100 while (m > 1 and m ≠ 4) do
  m := onestep(m);
   end do;
  if m = 1 then height := j; end if;
  height;
  end;
```

```
happy := proc(n)
    local m, j, height;
    m := n;
    height := -1;
    for j to 100 while 1 < m and m <> 4 do m := onestep(m) end do;
    if m = 1 then height := j end if;
    height
end proc                                                                    (4)
```

```
> for N from 2 to 100000 do if happy(3·N + 1) > 0 then if happy(3·N + 4) > 0 then
      if happy(3·N + 7) > 0 then print(3·N + 1, 3·N + 4, 3·N + 7) end if; end if; end if;
      end do;
```

```
                    169957, 169960, 169963
                    196957, 196960, 196963
                    199657, 199660, 199663                                  (5)
>
```

This naive search gives us the lowest sequence of three "3-consecutive" cubic happy numbers, 169957, 169960, 169963.

Now we look for four in a row, ie, with an arithmetic difference of 3. We write N as N1.d where d = d_1.d_0 is the last two digits.

Here a = S(N_1) and a+d_1^3 + d_0^3 = 1 mod 3, so d_0 = 1 - a - d1 mod 3.

```
> for a from 1 to 10000 do
   for d1 from 0 to 9 do
     s := (1 - a - d1) mod 3;
   for d0 from s to 9 by 3 do
   if happy(a + onestep(d1·10 + d0)) > 0 then if happy(a + onestep(d1·10 + d0 + 3)) > 0
       then if happy(a + onestep(d1·10 + d0 + 6)) > 0 then print(a, d1, d0, happy(a
        + onestep(d1·10 + d0 + 9))) end if; end if; end if;
   end do;
```

end do;
end do:

$$1341, 9, 4, -1$$
$$1675, 5, 7, -1$$
$$6350, 8, 6, -1$$
$$6854, 5, 6, 5$$
$$6854, 5, 9, -1$$
$$7043, 2, 9, -1$$
$$7062, 1, 6, 5$$
$$7062, 1, 9, -1$$
$$7463, 6, 8, -1$$
$$8581, 4, 5, 5$$
$$8581, 4, 8, -1$$
$$8642, 3, 8, -1$$
$$9252, 7, 6, -1 \quad \textbf{(6)}$$

We check to see if any of these extend to five in a row.

> $a := 6854;\ d1 := 5;\ d0 := 6;\ happy(a + onestep(d1 \cdot 10 + d0 + 12));$

$$a := 6854$$
$$d1 := 5$$
$$d0 := 6$$
$$-1 \quad \textbf{(7)}$$

> $a := 7062;\ d1 := 1;\ d0 := 6;\ happy(a + onestep(d1 \cdot 10 + d0 + 12));$

$$a := 7062$$
$$d1 := 1$$
$$d0 := 6$$
$$-1 \quad \textbf{(8)}$$

> $a := 8581;\ d1 := 4;\ d0 := 5;\ happy(a + onestep(d1 \cdot 10 + d0 + 12));$

$$a := 8581$$
$$d1 := 4$$
$$d0 := 5$$
$$-1 \quad \textbf{(9)}$$

Note there are not three in a row for digits 00, 01, 02 or 91, 92, 93, so if there are four in a row there cannot be a carry that is split three below 99 and one above 00, or one below 99 and three above 00. We use a separate Maple program LowCubic to find the best N associated with the three relevant n given in this first column. They are

n	L(n)
6854	668888999999
7062	4488888999999
8581	458888899999999

so the best is N = 66888899999956

>

> $N := 66888899999956;\ happy(N);\ happy(N + 3);\ happy(N + 6);\ happy(N + 9);\ happy(N$

$+ 12)$;

$$N := 66888899999956$$

6

6

6

6

-1 **(10)**

so this N does begin a 3-consecutive sequence of four cubic happy numbers.

Now we need to see if there could be a carry with the last two digits of N and N+3 at or before 99 and the last two digits of N+6 and N+9 at or after 00.
Note the last digit of N must be 4 or 5 or 6. Let k be the number of digits 9 at the end of N1 and let d2 <> 9 be the digit preceding these 9 digits, so N1 = N3.d2.999..99.

```
> for d0 from 4 to 6 do
  s := (1 - d0) mod 3;   # since a+9^3 + d0^3 = 1 mod 3, we have a = 1 - d0 mod 3.
  t := (6 + d0) mod 10;  # t will be the last digit of N+6.
  for a from s to 10000 by 3 do
  if happy(a + 9^3 + d0^3) > 0 and happy(a + 9^3 + (d0 + 3)^3) > 0 then
     ktop := iquo(a, 9^3);
    for k from 0 to ktop do  # k is the number of nines at the end of N1.
    for d2 from 0 to 8 do   # d2 is the digit that precedes the k digits of nine.
    if  happy( a + (d2 + 1)^3 - d2^3 - 9^3·k + t^3) > 0 then print(d0, a, k, d2, happy(a + a
      + (d2 + 1)^3 - d2^3 - 9^3·k + (t + 3)^3)) end if;
    end do; end do;
  end if;
  end do; end do:
```

4, 1341, 0, 0, -1

4, 1341, 0, 5, -1

4, 6798, 5, 4, -1

4, 6798, 6, 1, -1

5, 7916, 1, 1, -1

5, 7916, 1, 5, -1

5, 7916, 3, 4, -1

5, 8861, 10, 1, -1

5, 8861, 11, 7, -1

5, 8861, 12, 1, -1

5, 8957, 0, 8, -1

5, 8957, 2, 2, -1

5, 8957, 2, 5, -1

5, 8957, 10, 5, -1

5, 8957, 10, 6, -1

5, 8957, 11, 4, -1

5, 8957, 12, 0, -1

5, 9356, 3, 6, -1

5, 9356, 5, 1, −1
5, 9356, 9, 0, −1
5, 9356, 12, 7, −1
5, 9578, 1, 4, −1
5, 9578, 3, 6, −1
5, 9578, 5, 3, −1
5, 9578, 11, 2, −1
5, 9578, 11, 3, −1
5, 9578, 12, 7, −1
5, 9578, 13, 2, −1
6, 652, 0, 6, −1
6, 652, 0, 8, −1
6, 973, 0, 2, −1
6, 973, 0, 8, −1
6, 6133, 3, 7, −1
6, 6250, 3, 4, −1
6, 6250, 6, 1, −1
6, 6250, 6, 5, −1
6, 6250, 6, 6, −1
6, 6250, 6, 8, −1
6, 6250, 7, 7, −1
6, 6334, 6, 1, −1
6, 8866, 0, 3, −1
6, 8866, 9, 0, −1 **(11)**

So there are none where N and N+3 have last digits on or before 99 and N+6 and N+9 have last digits on or after 00.

Hence the minimal N starting a 3-consecutive sequence of four cubic happy numbers is given above.

We now ask about five cubic happy numbers in a 3-consecutive sequence. From the work above, any such must have the a>10000.

Note that our candidate, N= 35588899999799999999999999989 is a 29 digit number so we do not need to check any a>21141.

```
> evalf(log10(35588899999799999999999999989)); 9^3·29;
```

28.55131456

21141 **(12)**

```
> for a from 10000 to 15000 do
   for d1 from 0 to 9 do
     s := (1 − a − d1) mod 3;
   for d0 from s to 9 by 3 do
   if happy(a + onestep(d1·10 + d0)) > 0 and happy(a + onestep(d1·10 + d0 + 3)) > 0
       and happy(a + onestep(d1·10 + d0 + 6)) > 0 then print( a, d1, d0, happy(a
        + onestep(d1·10 + d0 + 9)), happy(a + onestep(d1·10 + d0 + 12))); end if;
   end do;
   end do;
```

end do:

$$10972, 9, 9, -1, 5$$
$$11458, 6, 6, -1, -1$$
$$11793, 5, 5, -1, -1$$
$$11854, 4, 5, 3, -1$$
$$11854, 4, 8, -1, -1$$
$$13018, 9, 0, -1, -1$$
$$13235, 8, 0, -1, -1$$
$$13342, 2, 7, -1, -1$$
$$13404, 6, 7, 3, -1$$
$$13404, 7, 0, -1, -1$$
$$13531, 6, 0, -1, -1$$
$$13622, 5, 0, -1, -1$$
$$13683, 4, 0, -1, -1$$
$$13691, 7, 7, -1, -1$$
$$13720, 3, 0, -1, -1$$
$$13739, 2, 0, -1, 6$$
$$13746, 1, 0, -1, -1$$
$$13747, 0, 0, -1, -1$$
$$13958, 1, 4, -1, -1$$
$$14131, 3, 9, -1, -1 \quad \textbf{(13)}$$

```
> for a from 15000 to 20000 do
   for d1 from 0 to 9 do
     s := (1 - a - d1) mod 3;
   for d0 from s to 9 by 3 do
   if happy(a + onestep(d1·10 + d0)) > 0 and happy(a + onestep(d1·10 + d0 + 3)) > 0
       and happy(a + onestep(d1·10 + d0 + 6)) > 0 then print( a, d1, d0, happy(a
       + onestep(d1·10 + d0 + 9)), happy(a + onestep(d1·10 + d0 + 12))); end if;
   end do;
   end do;
   end do:
```

$$16240, 8, 7, -1, -1$$
$$16342, 2, 7, -1, -1$$
$$16618, 8, 7, -1, -1$$
$$16736, 8, 9, 6, -1$$
$$16736, 9, 2, -1, -1$$
$$16850, 8, 6, -1, 4$$
$$16924, 1, 8, -1, -1$$
$$16953, 8, 2, -1, -1$$
$$16968, 2, 8, -1, -1$$
$$17029, 8, 7, -1, 4$$
$$17030, 6, 8, -1, -1$$
$$17122, 7, 2, -1, -1$$

17249, 6, 2, −1, −1
17309, 3, 8, −1, −1
17340, 5, 2, −1, −1
17372, 0, 8, −1, −1
17401, 4, 2, −1, 5
17438, 3, 2, −1, −1
17457, 2, 2, −1, −1
17464, 1, 2, 6, −1
17464, 1, 5, −1, −1
17465, 0, 2, −1, −1
17641, 5, 7, −1, −1
17749, 0, 9, −1, 4
17847, 0, 7, −1, 4
18019, 8, 7, −1, −1
18108, 9, 4, −1, −1
18264, 7, 9, −1, −1
18498, 7, 6, −1, −1
18718, 9, 0, −1, −1
18730, 2, 7, −1, −1
18777, 4, 6, −1, −1
18777, 8, 8, −1, −1
18810, 3, 4, 4, −1
18810, 3, 7, −1, −1
18846, 2, 5, −1, −1
18935, 8, 0, −1, −1
18956, 4, 7, −1, 4
19021, 9, 0, −1, −1
19022, 6, 5, −1, −1
19073, 8, 6, −1, 3
19104, 6, 7, 5, −1
19104, 7, 0, −1, −1
19124, 7, 7, −1, −1
19231, 6, 0, −1, −1
19238, 8, 0, −1, −1
19322, 5, 0, −1, −1
19383, 4, 0, −1, −1
19407, 6, 7, 5, −1
19407, 7, 0, −1, −1
19420, 3, 0, −1, −1
19439, 2, 0, −1, 4
19446, 0, 7, 5, −1
19446, 1, 0, −1, −1

$$19447, 0, 0, -1, -1$$
$$19498, 1, 8, -1, -1$$
$$19534, 6, 0, -1, -1$$
$$19625, 5, 0, -1, -1$$
$$19631, 0, 8, -1, -1$$
$$19635, 0, 7, -1, -1$$
$$19668, 0, 7, -1, -1$$
$$19686, 4, 0, -1, -1$$
$$19723, 3, 0, -1, -1$$
$$19742, 2, 0, -1, 6$$
$$19749, 1, 0, -1, -1$$
$$19750, 0, 0, -1, -1 \quad \textbf{(14)}$$

so there are none of length five without a carry.

We will try to extend the items above that could be a 3/2 or a 4/1 split over the carry. The only 4/1 possible split in this list is a=16736, and last digits 89, while the 3/2 splits are possible with last digits 00, 01, 02, namely a=13747, 17465, 19447, 19750, or at ones with last digits 91,92, or 93, of which there are none. First we look at a=16736 with last digits 89.

```
> a := 16736;
  ktop := iquo(a, 9^3);
  for k from 0 to ktop do  # k is the number of nines at the end of N1.
  for d2 from 0 to 8 do   # d2 is the digit that precedes the k digits of nine.
 if happy(a + (d2 + 1)^3 - d2^3 - 9^3·k + 1^3) > 0 then print(k, d2, happy(a + (d2 + 1)^3
     - d2^3 - 9^3·k
 + 4^3)) end if;
  end do; end do:
```

$$a := 16736$$
$$ktop := 22$$
$$13, 2, -1$$
$$13, 3, -1$$
$$15, 7, -1$$
$$19, 5, -1$$
$$21, 7, -1 \quad \textbf{(15)}$$

$> 16736 - 13\cdot 9^3 - 2^3;\ 16736 - 13\cdot 9^3 - 3^3;\ 16736 - 15\cdot 9^3 - 7^3;\ 16736 - 19\cdot 9^3 - 5^3;\ 16736 - 21\cdot 9^3 - 7^3;$

$$7251$$
$$7232$$
$$5458$$
$$2760$$
$$1084 \quad \textbf{(16)}$$

From the LowCubic program, the best N1 is for 5458 when N3 = 35588899999 So a candidate for N is
N = 35588899999.7.9999999999999999.89

> $N := 35588899999799999999999999989; happy(N); happy(N+3); happy(N+6); happy(N+9); happy(N+12); happy(N+15); happy(N-3);$

$$N := 35588899999799999999999999989$$
$$7$$
$$7$$
$$6$$
$$7$$
$$6$$
$$-1$$
$$-1 \quad (17)$$

>

We now check the others to verify they do not give a smaller value with five or more in a row.

```
> a := 13747;
    ktop := 1000;
  for k from 0 to ktop do #k is the number of 9s at the end of N1
  for d2 from 0 to 8 do  # d2 is the non-9 digit preceding the tail of 9s
  if happy(a - (d2+1)^3 + d2^3 + k·9^3 + 9^3 + 4^3) > 0 and happy(a - (d2+1)^3 + d2^3 + k
      ·9^3 + 9^3 + 7^3) > 0 then print(k, d2); end if;
  end do;
  end do:
```

$$a := 13747$$
$$ktop := 1000$$
$$410, 2$$
$$646, 1$$
$$772, 0 \quad (18)$$

> $13747 + 9^3 \cdot 410$

$$312637 \quad (19)$$

```
> a := 19447;
    ktop := 100;
  for k from 0 to ktop do #k is the number of 9s at the end of N1
  for d2 from 0 to 8 do  # d2 is the non-9 digit preceding the tail of 9s
  if happy(a - (d2+1)^3 + d2^3 + k·9^3 + 9^3 + 4^3) > 0 and happy(a - (d2+1)^3 + d2^3 + k
      ·9^3 + 9^3 + 7^3) > 0 then print(k, d2); end if;
  end do;
  end do:
```

$$a := 19447$$
$$ktop := 100$$
$$47, 1$$
$$76, 7 \quad (20)$$

> $19447 + 9^3 \cdot 47$

$$53710 \quad (21)$$

```
> a := 19750;
    ktop := 200;
  for k from 0 to ktop do #k is the number of 9s at the end of N1
```

```
for d2 from 0 to 8 do  # d2 is the non-9 digit preceding the tail of 9s
if happy(a − (d2 + 1)^3 + d2^3 + k·9^3 + 9^3 + 4^3) > 0 and happy(a − (d2 + 1)^3 + d2^3 + k
    ·9^3 + 9^3 + 7^3) > 0 then print(k, d2); end if;
end do;
end do:
```

$$a := 19750$$
$$ktop := 200$$
$$125, 0$$
$$152, 8$$
$$155, 6 \quad (22)$$

```
> 19750 + 9^3·125
```

$$110875 \quad (23)$$

```
> a := 17465;
   ktop := 100;
for k from 0 to ktop do #k is the number of 9s at the end of N1
for d2 from 0 to 8 do  # d2 is the non-9 digit preceding the tail of 9s
if happy(a − (d2 + 1)^3 + d2^3 + k·9^3 + 9^3 + 6^3) > 0 and happy(a − (d2 + 1)^3 + d2^3 + k
    ·9^3 + 9^3 + 9^3) > 0 then print(k, d2); end if;
end do;
end do:
```

$$a := 17465$$
$$ktop := 100$$
$$71, 5 \quad (24)$$

```
> 17465 + 9^3·71
```

$$69224 \quad (25)$$

```
>
```

so none of these give a smaller example.

```
> for a from 20000 to 22000 do
  for d1 from 0 to 9 do
    s := (1 − a − d1) mod 3;
  for d0 from s to 9 by 3 do
  if happy(a + onestep(d1·10 + d0)) > 0 and happy(a + onestep(d1·10 + d0 + 3)) > 0
      and happy(a + onestep(d1·10 + d0 + 6)) > 0 then print( a, d1, d0, happy(a
       + onestep(d1·10 + d0 + 9)), happy(a + onestep(d1·10 + d0 + 12))); end if;
  end do;
  end do;
  end do:
```

$$20305, 0, 9, -1, 3$$
$$20793, 5, 5, -1, -1 \quad (26)$$

so nothing here gives five in a row, hence our candidate above is indeed the best.

```
>
>
>
```