Smallest N Beginning a Sequence of Fourteen Consecutive Happy Numbers

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Abstract

It is well known that there exist arbitrarily long sequences of consecutive happy numbers. In this paper we find the smallest number beginning a sequence of fourteen consecutive happy numbers.

1 Introduction

In Richard Guy's *Unsolved Problems in Number Theory* [3], problem E34, a happy number is defined in the following way: "If you iterate the process of summing the squares of the decimal digits of a number, then it is easy to see that you either reach the cycle 4→16→37→58→89→145→42→20→4 or arrive at 1. In the latter case you started from a happy number." Written another way, a happy number N is one for which some iteration of the function $S(N) = \sum_{j=0}^{k} a_j^2$ returns a value of 1, where $N = \sum_{j=0}^{k} a_j 10^j$ is the decimal expansion of N. According to Guy, the problem was first brought to the attention of the Western mathematical world when Reg. Allenby's daughter returned with it from school in Britain. It is thought to have originated in Russia.

The first pair of consecutive happy numbers is 31, 32. The first example of three consecutive happy numbers is 1880, 1881, 1882. The smallest N's beginning a sequence of four and five consecutive happy numbers are 7839 and 44488, respectively. Siksek and El Sedy [1] were the first to publish a proof that there exist arbitrarily long sequences of happy numbers, although Lenstra is known to have had an unpublished proof before them. Styer [2] found the smallest examples of sequences of length *j* of consecutive happy numbers, for *j* from 6 to 13.

In this paper, we will use a period (.) to denote the concatenation operator to group sets of digits together within a large number. For convenience and clarity, we will also write large strings of 9's by their quantity in parenthesis. For example, $615\cdot 10^{157} + (10^{155}-1)\cdot 10^{2} + 71$ will be written as 615.(155 *nines*).71.

Define N_0 = 7888.(1604938271577 *nines*).1.(345696 *nines*).3.

2 Fourteen Consecutive Happy Numbers

Theorem

N_0 = 7888.(1604938271577 nines*).1.(345696* nines*).3 is the smallest N which begins a sequence of fourteen consecutive happy numbers.*

Note: N_0 has 1604938617279 digits.

Because the S function simply sums the squares of the digits of a number, and because addition is commutative, the ordering of the digits has no effect on the function's output. In other words,

Lemma 1. *S(A.B.C) = S(B.A.C) = S(A.C.B) = S(A) + S(B) + S(C).*

Lemma 2. *N_0 begins a sequence of fourteen consecutive happy numbers.*

Proof:

Before the carry	
N_0=7888.(1604938271577 nines).1.(345696 nines).3	S(N_0)= 130000027999364
N_0+1=7888.(1604938271577 nines).1.(345696 nines).4	S(N_0+1)= 130000027999371
N_0+2=7888.(1604938271577 nines).1.(345696 nines).5	S(N_0+2)= 130000027999380
N_0+3=7888.(1604938271577 nines).1.(345696 nines).6	S(N_0+3)= 130000027999391
N_0+4=7888.(1604938271577 nines).1.(345696 nines).7	S(N_0+4)= 130000027999404
N_0+5=7888.(1604938271577 nines).1.(345696 nines).8	S(N_0+5)= 130000027999419
N_0+6=7888.(1604938271577 nines).1.(345696 nines).9	S(N_0+6)= 130000027999436
After the carry	
N_0+7=7888.(1604938271577 nines).2.(345696 zeros).0	S(N_0+7)= 129999999997982
N_0+8=7888.(1604938271577 nines).2.(345696 zeros).1	S(N_0+8)= 129999999997983
N_0+9=7888.(1604938271577 nines).2.(345696 zeros).2	S(N_0+9)= 129999999997986
N_0+10=7888.(1604938271577 nines).2.(345696 zeros).3	S(N_0+10)= 129999999997991
N_0+11=7888.(1604938271577 nines).2.(345696 zeros).4	S(N_0+11)= 129999999997998
N_0+12=7888.(1604938271577 nines).2.(345696 zeros).5	S(N_0+12)= 129999999998007
N_0+13=7888.(1604938271577 nines).2.(345696 zeros).6	S(N_0+13)= 129999999998018

It is not difficult to see that each of these numbers is happy. The iterations of the S function get small rather quickly, and after at most nine steps, reach 1. **QED**

Lemma 3. *If* $N_a \le N_0$ *is another example of a number beginning a sequence of fourteen consecutive happy numbers, then* $S(N_a) \le 9^2 \cdot (1604938617279) = 130000027999599$.

Proof: In order for N_a to be smaller than N_0, it must not contain more digits than N_0. N_0 contains 1604938617279 digits. The largest number containing no more than 1604938617279 digits is $10^{1604938617279}-1$, or 1604938617279 digits 9, which has an S value of 130000027999599. Therefore if there were a number $N_a<N_0$ beginning a sequence of fourteen consecutive happy numbers, it would necessarily have an S value, $S(N_a) \le 130000027999599$. **QED**

We will let N_1 denote any candidate less its final digit. Thus we write, $N_a=N_1.x$, where x is the final digit. So in our case, $N_0=N_1.3$. Let d be the first (right-most) non-nine digit of N_1, and N_2 be the remaining digits of N_1, left of d. Thus we have

$$N_1 = N_2.d.(k \text{ nines}) \qquad (1)$$

for an integer $k \ge 0$.

Lemma 4. $S(N_1+1) \le S(N_1)+17$.

Proof:

$$N_1 = N_2.d.(k \text{ nines})$$

$$N_1+1 = N_2.(d+1).(k \text{ zeros})$$

$$S(N_1) = S(N_2) + d^2 + 81\cdot k$$

$$S(N_1+1) = S(N_2) + (d+1)^2$$

$$\begin{aligned} &= [S(N_2)+(d+1)^2] - (d+1)^2 + d^2 + 81\cdot k \\ &= S(N_1+1) - 2d - 1 + 81\cdot k \ge S(N_1+1) - 17 \end{aligned} \qquad d \le 8, k \ge 0$$

QED

Let M have four or more digits and m, f, g, h be integers. Define:

$$M = M_2.f.(m \text{ nines}).g.h \qquad (2)$$

where $m \ge 0$, and $0 \le g,f,h \le 8$.

Lemma 5. $S(M+e^2) = S(M_2) + S(f.(m \text{ nines}).g.h + e^2)$.

Proof: Since $e^2 \le 81$, then $g.h + e^2 \le 180$. Now $g.h + e^2 = i.j$ or $1.i.j$ for some digits *i* and *j*. Then we have $M + e^2 = M_2.f.(m \text{ nines}).i.j$ or $M_2.(f+1).(m \text{ zeros}).i.j$. Now Lemma 1 completes the argument. **QED**

Note that *130000027999599+17 = 130000027999616.*

Lemma 6. *If each member of the set* $\{M+e^2|e=2, 3, 4, 5, 6, 7, 8, 9\}$ *is happy, then* $M > 130000027999616$.

Proof: In [2], when dealing with fewer than fourteen consecutive happy numbers, Styer did an exhaustive search on all values of M to the needed bounds for his purposes. In order to reach a bound as high as 130000027999599, we ordered the digits of $M+e^2$. This made the search approximately seven million times more efficient.

Assume the digits of M_2 are ordered in non-decreasing order. For each *m* from 0 to 12, we have a separate Maple script that checks every possible M with M_2 ordered to see if every member of $\{M+e^2|e=2, 3, 4, 5, 6, 7, 8, 9\}$ is happy. Maple shows there are none. (For the relevant Maple scripts, see [4].) **QED**

A similar set of Maple calculations yields the following:

Lemma 7. *If each member of the set* $\{M+e^2|e=0, 1, 2, 3, 4, 5, 6, 7\}$ *is happy, then* $M>130000027999616$.

Lemma 8. *The value M=129999999997982 is the only M<130000027999616 such that every member of $\{M+e^2|e=0, 1, 2, 3, 4, 5, 6\}$ is a happy number. Let's call this value M_3.*

Proof: Maple calculations similar to Lemma 5 give this single example with digits in non-decreasing order. While any other permutation of the leading 11 digits (the M_2) will also result in every member of $\{M+e^2|e=0, 1, 2, 3, 4, 5, 6\}$ being a happy number, it will give us an M value which exceeds our bound.

QED

Lemma 9. *The value of $S(N_1)$ must satisfy $129999999997982-17 < S(N_1) < 130000027999599$.*

Lemma 10. *The only M with 129999999997982-17<M<130000027999599 such that every member of $\{M+e^2|e=3, 4, 5, 6, 7, 8, 9\}$ is a happy number is 130000027999355.*

A Maple search over all the numbers in the bound listed above returned this single result. Call this value M_1.

We now have the following relationships.

$$\begin{aligned} S(N_1) &= S(N_2) + d^2 + 81 \cdot k = 130000027999355 = M_1 \\ S(N_0 + 7) &= S(N_2) + (d+1)^2 = 129999999997982 = M_3 \\ M_1 - M_3 &= 81 \cdot k - 2d - 1 = 28001373 \end{aligned} \tag{3}$$

We look for integers k and d that satisfy this last relationship and get k = 345696 and d = 1.

Now all that's left is to find the smallest N_2 that will satisfy equations (3). With d = 1, it reduces to $S(N_2)$ = 129999999997978. Using the methods elaborated by Styer in [2], we easily find that the minimal N_2 for the above $S(N_2)$ is 7888.(1604938271577 *nines*). Putting all this together we see that the smallest N beginning a sequence of fourteen consecutive happy numbers is indeed N_0 = 7888.(1604938271577 *nines*).1.(345696 *nines*).3.

3 Fifteen Consecutive Happy Numbers

Using the same methods as outlined above, we have confirmed Styer's previous conjecture that the smallest number beginning a sequence of fifteen consecutive happy numbers is the following:

$$N=77.(2222222222222220).3.(97388).3\ .$$

References

[1] E. El-Sedy and S. Siksek, On happy numbers, *Rocky Mountain J. Math.*, **30** (2000), 565-570.

[2] R. Styer, Smallest Example of Strings of Consecutive Happy Numbers, *Journal of Integer Sequences*, **13** (2010), 10.6.3.

[3] R. Guy, *Unsolved Problems in Number Theory*, Spring-Verlag, 2nd ed., 1994.

[4] D. Lyons, Maple programs available at
`http://www.homepage.villanova.edu/robert.styer/HappyNumbers/happy_numbers.htm`