Practice Final Exam Math 1330 Fall 2006 ANSWERS

Please show all your work, including the main ideas and the main algebra steps. And please *relax*, **RELAX**, relax...

1. Consider light bulbs which generally have an exponential probability distribution, ce^{-t/t_0} for some constant c.

(a.) Set up the integral and equation one would use to find the constant c.

$$1 = \int_0^\infty ce^{-t/t_0}\, dt$$

(b.) Solve this integral and determine c.

$$1 = \int_0^\infty ce^{-t/t_0}\, dt = c\int_0^\infty e^{-t/t_0}\, dt$$

Using u-substitution, one finds the antiderivative is $-t_0 e^{-t/t_0}$ and at ∞ this is zero so the integral becomes

$$1 = \int_0^\infty ce^{-t/t_0}\, dt = ct_0$$

hence $c = t_0$.

2. Halogen bulbs often have a listed (mean) lifetime about 2000 hours. The probability distribution for these bulbs is

$$f(t) = \frac{1}{2000}e^{-t/2000}$$

and next year you will see that the mean is calculated by using the integral

$$\int_{t=0}^{t=\infty} tf(t)\, dt$$

(a.) Use integration by parts with $u = t$ to calculate this integral.

Use $u = t$ and $dv = f(t)\, dt$ and thus $du = dt$ and $v = -e^{-t/2000}$ to get

$$\int_{t=0}^{t=\infty} tf(t)\, dt = -te^{-t/2000}|_0^\infty - \int_0^\infty (-e^{-t/2000})\, dt = (0-0) + \int_0^\infty e^{-t/2000}\, dt = -2000e^{-t/2000}|_0^\infty = 2000$$

(b.) Now calculate the median. Explain why these are not the same, and why the median is smaller than the mean.

Let the median value be M. Then

$$\int_{t=0}^{t=M} f(t)\, dt = 0.50$$

hence after taking out the constant and using a u-substitution,

$$\int_0^M \frac{1}{2000}e^{-t/2000})\, dt = -e^{-t/2000}|_0^M = -e^{-M/2000} - (-e^{-0/2000}) = 1 - e^{-M/2000} = 0.50$$

and now use algebra to get $0.50 = e^{-M/2000}$ and taking logs of both sides, $\ln(0.50) = -M/2000$ so $M = -2000\ln(-0.50)$ hence

$$M = 2000\ln(2) = 1386.3$$

We note that the median is substantially less than the mean. This is because the distribution is compact on the left of the median and spread out far to the right of the median, hence, the pivot point (which is the mean) would be to the right of the median.

3. A distribution that arises in analysis of binomial trials (flipping coins, a stock increasing or decreasing, a company making a profit or not) is the beta distribution, $B(x) = c * x^a(1-x)^b$ where a and b depend on the particular problem, and where the domain is $0 \le x \le 1$. Assume $a = 2$ and $b = 1$, so $B(x) = cx^2(1-x)$.

(a.) Calculate the value of c that makes this a valid probability, i.e., find $\int_0^1 cx^2(1-x)\,dx = 1$. HINT: the answer is $c = 12$.

$$1 = \int_0^1 cx^2(1-x)\,dx = c\int_0^1 (x^2 - x^3)\,dx = c(\frac{x^3}{3} - \frac{x^4}{4})|_0^1 = c(\frac{1}{3} - \frac{1}{4}) = c/12$$

hence $c = 12$.

(b.) Find the mode of this distribution.

The mode is the maximum value on the domain. The endpoints of 0 and 1 have height zero, so are minima not maxima. Hence, the max is at a critical point.

$$B'(x) = c(x^2 - x^3)' = c(2x - 3x^2) = cx(2 - 3x) = 0$$

and so $x = 0$ or $x = 2/3$ and we already argued that $x = 0$ is not the max hence the max is when $x = 2/3$, that is, the mode is when $x = 2/3$.

(c.) Find the median of this distribution.

Let M be the median.

$$0.50 = \int_0^M 12x^2(1-x)\,dx = 12\int_0^M (x^2 - x^3)\,dx = 12(x^3/3 - x^4/4)|_0^M = 4M^3 - 3M^4$$

Thus, we need to solve the equation $3M^4 - 4M^3 + 0/5 = 0$, which using my TI calculator Solver button or Calc Root button or zooming in on the axis to see where is crosses gives $M = 0.61427243186$.

(d.) The mean is given by

$$\int_{x=0}^{x=1} xB(x)\,dx = \int_{x=0}^{x=1} 12x^3(1-x)\,dx$$

Calculate this mean.

$$\int_{x=0}^{x=1} 12x^3(1-x)\,dx = 12\int_0^1 (x^3 - x^4)\,dx = 12\left(\frac{x^4}{4} - \frac{x^5}{5}\right)|_0^1 = 12(1/4 - 1/5) = 12/20 = 0.6$$

(e.) Explain why the mean is smaller than the median.

Looking at the graph of the function, it is compact to the right of the median and stretched out to the left of the median, hence the pivot point must be to the left of the median.

4. Find these integrals

(a.) $\int_0^2 \frac{x^2+3}{5\sqrt{x}}\, dx$

Algebra converts this product to a sum!

$$\int_0^2 \frac{x^2+3}{5\sqrt{x}}\, dx = \frac{1}{5}\int_0^2 (x^{3/2} + 3x^{-1/2})\, dx = \frac{1}{5}(\frac{2}{5}x^{5/2} + 6x^{1/2})|_0^2 = \frac{38\sqrt{2}}{25} = 2.15$$

(b.) $\int e^x(1+x)\, dx$

Algebra, then the second integral is integration by parts with $u = x$ and $dv = e^x\, dx$ hence $du = dx$ and $v = e^x$:

$$\int e^x(1+x)\, dx = \int e^x\, dx + \int xe^x\, dx = e^x + C_1 + xe^x - \int e^x\, dx = e^x + xe^x - e^x + C = xe^x + C$$

(c.) $\int_1^1 6x^{3/4}\, dx$

The integral is zero, since it begins and ends the same place.

(d.) $\int_{10}^{30}(-0.00345x^3 + 0.00456x^2 + 0.0789x - 9.123)\, dx$

Use the Sum Rule and the Multiplicative Constant Rule to get

$$\int_{10}^{30}(-0.00345x^3 + 0.00456x^2 + 0.0789x - 9.123)\, dx =$$

$$-0.00345\int_{10}^{30} x^3\, dx + 0.00456\int_{10}^{30} x^2\, dx + 0.0789\int_{10}^{30} x\, dx - 9.123\int_{10}^{30} dx =$$

$$-0.00345x^4/4|_{10}^{30} + 0.00456x^3/3|_{10}^{30} + 0.0789x^2/2|_{10}^{30} - 9.123x|_{10}^{30} =$$

$$-690 + 39.52 + 31.56 - 182.46 = -801.38$$

(e.) $\int 3x^2e^{-x^3}\, dx$

U-substitution, $u = -x^3$, so $du = -3x^2\, dx$ so $-du = 3x^2\, dx$

$$\int 3x^2e^{-x^3}\, dx = \int e^u(-du) = -e^u + C = -e^{-x^3} + C$$

(f.) $\int z\sqrt{5z-1}\, dz$

U-substitution, $u = 5z - 1$ so $du = 5dz$ so $dz = du/5$. But we also need z in terms of u so note that $z = (u+1)/5$:

$$\int z\sqrt{5z-1}\, dz = \int \frac{u+1}{5}\sqrt{u}\frac{du}{5} = \frac{1}{25}\int (u+1)\sqrt{u}\, du = \frac{1}{25}\int (u^{3/2} + u^{1/2})\, du =$$

$$\frac{1}{25}\left(\frac{u^{5/2}}{5/2}+\frac{u^{3/2}}{3/2}\right)+C=\frac{2}{125}u^2\sqrt{u}+\frac{2}{75}u\sqrt{u}+C=\left(\frac{2}{125}(5z-1)+\frac{2}{75}\right)(5z-1)\sqrt{5z-1}+C$$

(g.) $\int_{r=2}^{r=4}\frac{r^2-3}{r}\,dr$

$$\int_{r=2}^{r=4}\frac{r^2-3}{r}\,dr=\int_{r=2}^{r=4}\left(r-\frac{3}{r}\right)dr=r^2/2|_2^4-3\ln(r)|_2^4=(8-2)-3(\ln(4)-\ln(2)=6-3\ln(2)=3.92056$$

(h.) $\int_{r=2}^{r=4}\frac{r}{r^2-3}\,dr$

U-substitution, $u=r^2-3$ so $du=2r\,dr$ so $r\,dr=du/2$

$$\int_{r=2}^{r=4}\frac{r}{r^2-3}\,dr=\int_{u=1}^{u=13}u^{-1}\frac{du}{2}=\frac{\ln(u)}{2}|_{u=1}^{u=13}=\ln(13)/2=1.28247$$

(i.) $\int_1^{\infty}\frac{1}{x^5}\,dx$

$$\int_1^{\infty}\frac{1}{x^5}\,dx=\lim_{N\to\infty}\int_1^N\frac{1}{x^5}\,dx=\lim_{N\to\infty}\frac{x^{-4}}{-4}|_{x=1}^{x=N}=\lim_{N\to\infty}\frac{1}{-4N^4}-\frac{1}{-4}=0+\frac{1}{4}=\frac{1}{4}$$

BOOK HAS ODD NUMBERED ANSWERS IN BACK.

5. Section 4.6 Problems 1-28, and 4.8 Problems 5 and 6.

6. (a.) Section 4.7 Problems 1-6

(b.) Section 4.7 Problems 7-9 but separating out the Modeling as one problem and the Surplus as a separate problem.

ANSWERS TO PROBLEM LIKE 7 AND 8 SEPARATE SHEET.

7. The standard normal probability distribution is also known as the bell curve. Suppose SAT scores are normally distributed. A famous principle in business is the Six Sigma Rule. To calculate the probability that a normalized score is in this six sigma region, i.e., within three standard deviations, we can calculate the integral

$$\int_{z=-3}^{z=3}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz$$

But there is no elementary antiderivative for this integrand, so one must use numerical methods.

(a.) Draw a picture of the trapezoids one would use for the Trapezoidal Rule.

(b.) Find the area of each of the three trapezoids one would use for the Trapezoidal Rule with $n=3$ boxes.

(c.) Sum these to find the Trapezoidal Rule estimate, and compare to the exact answer calculated on your calculator. What is the absolute error and what is the relative error?

(d.) Repeat (a.)-(c.) with the Midpoint Rule.

(e.) Using your answers to the Trapezoid and Midpoint Rule, find the Simpson's Rule approximation with $n = 3$ boxes. Compare the Simpson's Rule answer to the exact answer calculated on your calculator. What is the absolute error and what is the relative error?

8. Same as Problem 7 but use the integral $\int_{x=0}^{x=1} x^{0.2345}(1-x)^{0.6789}\,dx$ with $n = 4$ boxes.

9. In mathematical ecology, the Ricker model describes the relationship between the size of the parental stock of some fish and the number of recruits. If we denote the size of the parental stock by N and the number of recruits by x, then the Ricker curve is given by $N(x)$ where A and B are positive constants:

$$N(x) = \frac{x}{A} e^{-x/B}$$

Find the maximal value of N using algebraic calculus methods, not technology.

The maximal value cannot occur at the endpoint $x = 0$ which is a minimum, since it is zero when $x = 0$ and is positive for all $x > 0$. So we need to find the critical points. We use the product rule: $0 = N'(x) = \frac{1}{A}e^{-x/B} + \frac{x}{A}e^{-x/B}\frac{-1}{B} = \frac{1}{A}e^{-x/B}(1 - x/B)$ but the first two factors can never be zero hence the third must be the zero factor: $1 - x/B = 0$ hence $x = B$ is the critical point, and thus, the maximum is when $x = B$.

10. Calculate the derivative of each of these functions:

(a.) $\frac{1}{\sqrt{9+x^2}}$ Chain Rule: $\frac{-x}{(9+x^2)\sqrt{9+x^2}}$

(b.) $(3t+2)e^{-t^2/2}$ Product Rule: $3e^{-t^2/2} + (3t+2)e^{-t^2/2}(-t) = (3 - 2t - 3t^2)e^{-t^2/2}$

(c.) $\ln(3x+17) + \frac{1}{3x+17} - \ln(3\pi+17)$ Sum Rule and Chain Rule and Additive Constant Rule: $\frac{3}{3x+17} - \frac{3}{(3x+17)^2}$

(d.) $\frac{\ln(q)}{5q}$ Mult. Const. Rule and Quotient Rule: $\frac{1}{5}\frac{q/q - q\ln(q)}{q^2} = \frac{1-\ln(q)}{5q^2}$

(e.) $\frac{x}{x^2+x+1}$ Quotient Rule: $\frac{(x^2+x+1)-x(2x+1)}{(x^2+x+1)^2} = \frac{1-x^2}{(x^2+x+1)^2}$

(f.) $\frac{x^2+x+1}{x}$ Note that this one is done by algebra NOT the quotient rule: $1 - 1/x^2$

(g.) $\frac{3}{x^2+x+1}$ Thinl Power, then Chain Rule: $\frac{-3(2x+1)}{(x^2+x+1)^2}$

(h.) $\frac{x^2+x+1}{3}$ Note that this one NOT done by the quotient rule: $\frac{2x+1}{3}$

(i.) $\frac{\pi^2+\pi+1}{x}$ Note that this one NOT done by the quotient rule: $\frac{-(\pi^2+\pi+1)}{x^2}$

(j.) $\frac{\pi^2+\pi+1}{3}$ Note that this is a Constant! zo derivative is Zero.

11. Use the limit definition of derivative and show all algebra steps to find the derivative of:

$$\frac{3}{x}, \frac{1}{x^2}, 5x^2, 5x-3, \frac{x^3}{3}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0}\frac{\frac{3}{x+h} - \frac{3}{x}}{h} = \lim_{h\to 0} 3\frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} =$$

$$\lim_{h\to 0} 3\frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} 3\frac{\frac{-h}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-3}{x(x+h)} = \frac{-3}{x(x+0)} = \frac{-3}{x^2}$$

Next function:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{-2xh-h^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} = \frac{-2x-0}{x^2(x+0)^2} = \frac{-2}{x^3}$$

Next function:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{5(x+h)^2-5x^2}{h} =$$

$$\lim_{h\to 0} 5\frac{(x^2+2xh+h^2)-x^2}{h} = \lim_{h\to 0} 5\frac{2xh+h^2}{h} = \lim_{h\to 0} 5(2x+h) = 5(2x+0) = 10x$$

Next function:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{(5(x+h)-3)-(5x-3)}{h} = \lim_{h\to 0} \frac{5h}{h} = 5$$

Next function:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h\to 0} \frac{\frac{(x+h)^3}{3} - \frac{x^3}{3}}{h} = \lim_{h\to 0} \frac{1}{3}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0} \frac{1}{3}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}$$

$$= \lim_{h\to 0} \frac{1}{3}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} \frac{1}{3}(3x^2+3xh+h^2) = \frac{1}{3}(3x^2+3x\cdot 0+0^2) = x^2$$

ANSWERS SEPARATE SHEET

12. Find the derivative dy/dx numerically using a table as we did in class, with at least four orders of magnitude, for these functions at the point $t = 3$: $y = 3t^2$, $y = 3^t$, $y = 3/t$, $y = 3/t^2$.

ANSWERS 13 AND 14 SEPARATE

13. For each of the graphs given on the separate page, draw the tangent line at $t = 2$ and find the slope of the line graphically. Please show each idea and step.

14. For each of the graphs given on the separate page, draw the corresponding slope graph, making sure to correspond the important points on the two graphs.

15. Lay out the argument as in the text and as in class that the average cost is minimized when the average cost equals the marginal cost.

HINT: You should argue as follows:

Let $C(q)$ be the total cost to manufacture quantity q. Then the average cost $A(q) = C(q)/q$ by definition of average cost.

Clearly, $A(q)$ is not minimized at the endpoints $q = 0$ or $q = \infty$. Hence, the average cost is minimized when the derivative is zero: $A'(q) = 0$.

Now $A(q)$ is a quotient so we use the Quotient Rule:

$$A'(q) = 0 \rightarrow \frac{C'(q) \cdot q - C(q) \cdot 1}{q^2} = 0$$

$$C'(q) \cdot q - C(q) \cdot 1 = 0$$

$$C'(q) \cdot q = C(q)$$

$$C'(q) = \frac{C(q)}{q}$$

$$C'(q) = A(q)$$

since $A(q) = C(q)/q$ by definition of average cost, and $C'(q)$ is the marginal cost by definition.

We conclude that when the average cost is minimized, the average cost equals the marginal cost.

(Side comment: this is also clear from common sense, but our goal here is to see how one can verify a common-sense idea mathematically.)