Midterm Exam Algebraic Number Theory, Summer 2005

Rules: This is an open textbook, open notes, open mind, but not open-mouth, open world-wide-web search, open library search, or open tutoring service. Thus, you can ask anyone anything about any homework problem that is not on the exam, but ask no one (except me) anything about the exam problems. You may quote any results in our text up to the problem in question.

1. Explictly find a polynomial that shows that $(1+\sqrt{-7})/2$ is an algebraic integer. Then show that $(1+\sqrt{7})/2$ is not an algebraic integer.

2. Define "irreducible" and "prime." Give an example of a quadratic field where these two notions are distinct, and then give an example of a quadratic field where these two notions coincide (and quote the result that guarantees this for your example.)

3. In $D = \boldsymbol{Z} + \boldsymbol{Z}\sqrt{-6}$ show that $< 2, \sqrt{-6} >$ is not equal to D nor to $< 2 >$. Show that $< 7, 1+\sqrt{-6} > \neq < 7, 1-\sqrt{-6} >$.

4. Theorem 2.1.2 is important and its proof is paradigmatic. Study this proof carefully, then with your book and notes closed, write out a proof of this theorem.

5. Prove Exercise 3.7.

6. Prove Exercise 3.8.

7. Let us explore the simplest case of intersections and sums of submodules. Let $R = \boldsymbol{Z}$ and consider the ideals $M_1 =< 4 >$, $M_2 =< 5 >$, $M_3 =< 6 >$, and $M_4 =< 10 >$.
(a.) Find $M_i \cap M_j$ for all six possibilities of i and j, $j > i$.
(b.) Find $M_i + M_j$ for all six possibilities of i and j, $j > i$.
(c.) For general rational integers a and b, the intersection $< a > \cap < b >$ yields what principal ideal? the sum $< a > + < b >$ yields what principal ideal?

8. Exercise 5.7

9. Exercise 5.13 (may use Maple if you wish.)

10. Exercise 5.20

11. Exercise 6.7

12. Exercise 6.17 In light of this proof, what is one reason the square in the discriminant is important?