A SPECTRAL SEQUENCE CONNECTING
THE SPENCER COHOMOLOGY
OF A TRANSITIVE LIE ALGEBRA WITH ITS
DEFORMATION COHOMOLOGY

Klaus Volpert
Department of Mathematical Sciences
Villanova University, Villanova, PA 19085

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Abstract

The notion of transitive (or complete filtered) Lie algebra was introduced in 1964 by Guillemin and Sternberg and examined by Kobayashi and Nagano as an algebraic model to study transitive differential geometries and their pseudogroups.

Any such filtered Lie algebra \( L \) gives rise to a graded Lie algebra \( G \). The graded algebra can be interpreted to represent the "flat" geometry, while the filtered algebra can be viewed as a "one-parameter-deformation" of that geometry. It is an interesting problem now to find and classify all filtered Lie algebras that have a given graded (transitive) algebra \( G \) as their associated graded algebra.

The key to such a classification lies in the cohomology theory of \( G \). Partial results are known in terms of two different cohomologies: the "Spencer" cohomology, studied by geometrists, and the "positive" cohomology, studied by algebraists in the context of "infinitesimal" deformations of algebras.

In this paper we will describe explicitly a spectral sequence which shows certain Spencer elements to be the "asens" of the positive cohomology. That allows us to generalize to arbitrary order known first- and second-order criteria for the "non-deformability" of \( G \). Finally it entails an easy proof that the positive cohomology of a transitive Lie algebra is always finite-dimensional and that therefore there are only finitely many different filtered Lie algebras with the same graded Lie algebra (up to a certain equivalence).

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1. The spectral sequence of a doubly graded complex.

A sequence of differential complexes \( \{E_r, d_r\} \) in which each \( E_r \) is the cohomology of its predecessor \( E_{r-1} \) is called a spectral sequence. If \( E_r \) eventually becomes stationary, then the stationary value is denoted by \( E_{\infty} \). If \( E_{\infty} \) is isomorphic to the associated graded group of some filtered group \( H \), then the spectral sequence is said to converge to \( H \).

We will first prove a general theorem on spectral sequences, which makes it possible to express all differential operators \( d_r \) explicitly on the chainlevel, which in turn enables us to calculate the limit cohomology without any of the usual degeneracy conditions. This is a generalization of the so-called "tic-tac-toe scheme" that Bost and Tu [1] invented for a differential operator \( D = d_3 + (-1)^3 d_3 \).

So suppose \( K = \prod_{i \geq 0} K^i \) is a graded vectorspace over \( \mathbb{R} \) or \( \mathbb{C} \) with differential operator \( D : K^i \to K^{i+1} \). Suppose furthermore that there is another grading \( K^i = \prod_{j \geq 0} K^{i,j} \). We say that \( D \) is bounded from the left, if there exists an \( m \in \mathbb{Z} \) such that whenever \( f \in K^{i,j} \) then \( Df \in K^{i+m,j+1} \oplus K^{i+m,j+1} \oplus K^{i+m+2,j+1} \oplus \cdots \).

The largest such \( m \) we will call the shift of \( D \) with respect to this grading. We can then write

\[
D = d_m + d_{m+1} + d_{m+2} + \cdots
\]

where \( d_j \) is the image of \( Df \) in \( K^{i+m,j+1} \).

\[
\begin{array}{cccccc}
K^{i+m,j+k} & K^{i+m+1,j+k} & K^{i+m+2,j+k} & K^{i+m+3,j+k} & \cdots \\
| & d_m & | & d_{m+1} & | & d_{m+2} & | & d_{m+3} & |
\end{array}
\]
We denote with \((Df)\), the truncated part of \(Df\) that lies in \(K^{i,j} @ K^{i,j} @ K^{i,j} @ \ldots @ K^{i,j}\).

Theorem 1.1. Let \(K = \prod_{i,j} K^{i,j}\) be a doubly graded complex with differential operator \(D: K^i \to K^{i+1}\). Suppose \(D\) is bounded from the left with shift \(m\). Write \(D = d_m + d_{m+1} + d_{m+2} + \cdots + d_i + \cdots\), where \(d_i : K^{i,j} \to K^{i+1,j+1}\). Then there exists a spectral sequence \(\{d_r : E_r^{i,j} = E_r^{i,j} @ E_r^{i,j} @ \ldots @ E_r^{i,j} \mid r \geq m\}\) with \(E_1 = H^0_d(E_{0,1})\) and \(E_m = K\) such that

(a) \(f \in K^{i,j}\) represents an element \([f] \in E_1^{i,j}\) if there exists an extension \(\text{ext}_r f = f + f_1 + f_2 + \cdots + f_{r-1}\) such that \(d_r[f] = 0\).

(b) \([f] = 0\) in \(E_1^{i,j}\) if there exists an extension \(\text{ext}_r g = g + g_1 + \cdots + g_{r-1}\) such that \(D(\text{ext}_r g) = 0\) and \(D(\text{ext}_r g) = f\).

(c) \(d_{r+1}[f] = D(\text{ext}_r f) + f_{r+1} = d_m f + d_{i+1} f + \cdots + d_{m+2} f_{r-1} + d_{m+1} f_{r-1}\) defines a differential operator \(\partial_{r+1} : E_r^{i,j} \to E_r^{i,j+1} @ \ldots @ E_r^{i,j}\) independent of the choice of the particular extension of \(f\).

For example, suppose that \(m = 0\) and \(D = d_4 + d_4 + d_4 + d_4 + \cdots\). Then \(d_4 : K^{i,j} \to K^{i,j+1}\) is the first differential operator of the spectral sequence and an element \([f] \in E_1^{i,j}\) is represented by an element \(f\) in \(K^{i,j}\) such that \(d_4 f = 0\). The next differential \(d_5\) is given by \(d_5[f] = [d_4 f]\).
The same \( f \) represents an element of \( E_k^i \), if there exists some \( g \) in \( K \) such that \( d_ig = d_if \) and the differential \( \partial_i \) is given by \( d_if - d_ig \).

Staying with the choice of \( g \), \( f \) represents an element of \( E_k^{i+1} \), if there exists some \( h_1, h_2 \) with \( d_1h_1 = 0 \) such that \( d_1f - d_1g = d_1h_1 + d_2h_2 \) and the next differential \( \partial_i \) is given by \( d_1f - d_1(g + h_1) - d_2h_2 \) and so on.

Proof of Theorem 1.1 by induction on \( r \). We set \( m = 0 \) for ease of notation.

(i) \( d_1[f] \) defines an element in \( E_2^{i+1+j+1} \), we have to show that \( D(\text{ext}_r f)_{i+j+1} \) as an element in \( K^{i+j} (r+1) \) has an extension \( \text{ext}_{r+1} (D(\text{ext}_r f)_{i+j+1}) \) so that its image under \( D \) is zero mod \( K^{i+j+1} (r+1) \). But \( D(\text{ext}_r f)_{i+j+1} \) is such an extension, since \( D^2 = 0 \).
(ii) The definition of $\mathcal{D}_{k+1}[f]$ does not depend on the extension of $f$ chosen: suppose
\[ \text{ext}_f f = f + f_1 + f_2 + \cdots + f_r \]
\[ \text{ext}_f f = f + f_1 + f_2 + \cdots + f_r \]
induce the same element $[f] \in \mathbb{E}^j$. Then there exists an element $[g] \in \mathbb{E}^{j+1}$ an extension $\text{ext}_g g = g + g_1 + g_2 + \cdots + g_r$ such that
\[ D(\text{ext}_g g) = 0 \]
\[ D(\text{ext}_g g) = f - f. \]

We need to find an extension $\text{ext}_h h = h + h_1 + \cdots + h_r$ such that $D(\text{ext}_h h) = 0$ and $D(\text{ext}_h h)_{i+1} = D(\text{ext}_h f - \text{ext}_h f)_{i+1}$.

But notice that $\text{ext}_h h := \text{ext}_f f - \text{ext}_f f - D(\text{ext}_h g)_{i+1}$ works.

(iii) By part (i) we know that $D(\text{ext}_f f)_{i+1}$ is an extension for $D(\text{ext}_f f)_{i+1}$, so
\[ \mathcal{D}_{k+1}[f] = \mathcal{D}_{k+1}[D(\text{ext}_f f)_{i+1}] \]
\[ = D(\text{ext}_f D(\text{ext}_f f)_{i+1})_{i+2} \]
\[ = D(D(\text{ext}_f f)_{i+2})_{i+3} \]
\[ = d_2(D(\text{ext}_f f) - D(\text{ext}_f f))_{i+1} \]
\[ = 0 \text{ in } \mathbb{E}. \]

(iv) Suppose now that $[\cdot]$ represents an element of $\mathbb{E}^{j+1}_{k+1}$, i.e., $\mathcal{D}_{k+1}[f] = 0$ in $\mathbb{E}^{j+1}_{k+1}$. Then there exists by induction hypothesis an $[h] \in \mathbb{E}^{j+1}_{k+1}$ and an extension $\text{ext}_h h = h + h_1 + \cdots + h_r$ such that $D(\text{ext}_h h) = 0$ and $D(\text{ext}_h h)_{i+1} = D(\text{ext}_f f)_{i+1}$. Notice then that $\text{ext}_h h := \text{ext}_f f - \text{ext}_h h$ is an extension such that $D(\text{ext}_h h)_{i+1} = 0$. 

The converse direction is easy to see. Suppose \( \text{ext}_{c+1} f = f + f_1 + \cdots + f_r + f_{r+1} \)

is an \((r+1)\)-extension of \( f \) such that \( D(\text{ext}_{c+1} f)|_{E^i} = 2 \). Then \( \text{ext}_c f = f + f_1 + \cdots + f_r \) is an \( r \)-extension such that \( D(\text{ext}_c f)|_{E^i} = 2 \) and by induction \( [f] \in E^i_{E^i} \).

Now notice that \( D(\text{ext}_c f)|_{E^i} = 2 \). Hence \( \hat{D}_{c+1} f = 0 \) in \( E^{i+r+1}_{E^i} \) and \([f]\) defines an element of \( E^{i+r+1}_{E^i} \).

(v) Finally, suppose that \([f]\) is 0 in \( E^{i}_{E^i} \). Then there exists \([g]\) in \( E^{i+r-1}_{E^{i-r}} \) such that \([f] = \hat{D}_{c+1} [g] \). Let \( \text{ext}_{c+1} g = g + g_1 + \cdots + g_r \) be an \( [g]\)-extension of \( g \) such that \( D(\text{ext}_{c+1} g)|_{E^i} = 0 \) and \([f] - D(\text{ext}_{c+1} g)|_{E^i} = 0 \) in \( E^{i}_{E^i} \).

By induction hypothesis, there exists an \([h]\) in \( K^{i-r+1}_{i-r} \) and an \( \text{ext}_{c+1} h = h + h_1 + \cdots + h_r \) such that \( D(\text{ext}_{c+1} h)|_{E^i} = 0 \) and \( D(\text{ext}_{c+1} h)|_{E^i} = f - D(\text{ext}_{c+1} g)|_{E^i} \).

Define \( \text{ext}_{c+1} : = \text{ext}_{c+1} g + \text{ext}_{c+1} h \)

\( \quad = g + (g_1 + h_1) + (g_2 + h_2) + \cdots + (g_r + h_{r-1}) + h_r \)

then \( D(\text{ext}_{c+1} f)|_{E^i} = 0 \) and \( D(\text{ext}_{c+1} f)|_{E^i} = f \), which we had to show.

The converse is true by definition of \( E^{i}_{E^i} \).

It will turn out that the sequences we will deal with have the following crucial property: from a certain leaf on, the \( E^{i}_{E^i} \) are bounded. That means that for every \( f \) there exists a smallest integer \( s_j \) such that \( E^{i}_{E^i} = 0 \) for all \( i \) greater or equal to \( s_j \). Let us call \( s = \max(s_1, s_2, s_3) \) the order of \( K \) and the numbers \( i > s \) the stable range of \( K \).

Since, by hypothesis, \( E^{i}_{E^i} = 0 \) for \( i \leq 0 \), we have then that for \( r \geq \max(s_1, s_2, s_3) \) both maps

\[ d_{c+1} : E^{i+r-1}_{E^i} \to E^{i}_{E^i} \quad \text{and} \quad d_{c+1} : E^{i}_{E^i} \to E^{i+r+1}_{E^i} \]
are zero. That implies that on each edge \( j \) the spectral sequence stabilizes after a finite number of steps or levels. These stable groups are usually denoted by \( E_\infty^{j,k} \).

For our purposes it will suffice to consider the case when the spectral sequence is bounded after the first level \( E_1 \). In that case we can work with the fact that whenever \( f \in K^{k,j} \) and \( i \) is in the stable range and \( df = 0 \) (in particular, when \( f \) induces an element \( \{f\} \) in \( E_\infty^{j,i} \)) then there exists \( g \in K^{k,j+1} \) such that \( f = dg \).

Proposition 1.2. If \( f \) defines an element \( \{f\} \) in \( E_\infty^{j,i} \), then there exists a sequence of extensions \( \{\text{ext}_j f, r \geq 1\} \) with \( D(\text{ext}_j f)_{(r+1)} = 0 \) such that \( \text{ext}_j f - \text{ext}_{j-1} f \in K^{r+r+1} \) whenever \( r \) is in the stable range. Hence the limit \( \text{ext}_j f = \lim_{r \to \infty} \text{ext}_j f \) exists and \( D(\text{ext}_j f) = 0 \).

Proof. Extensions exist up to arbitrary order by Theorem 1.1. Once \( r \) is in the stable range, the next extension can be obtained from the previous one by just adding a higher order term.

Since \( d_0(D(\text{ext}_j f)_{(r+1)}) = 0 \), there exists \( g \in K^{r+r+1} \) such that \( dg = D(\text{ext}_j f)_{(r+1)} \). The new extension \( \text{ext}_{r+1} f := \text{ext}_j f - g \) differs from the previous one only by an element in \( K^{r+r+1} \) and \( D(\text{ext}_{r+1} f) = 0 \) up to order \( i+r+1 \).

It is clear then that the limit for \( r \to \infty \) is a cocycle of \( D \).

Let us look at an element \( f \) in \( K^{k,j} \). We know after a finite number of steps whether \( f \) defines an element \( \{f\} \) in \( E_\infty^{j,i} \). Along the way we obtain a sequence of extensions of \( f \) which stabilizes after a while and in the limit defines an element of \( H_0^j(K) \). How unique is this element?

One can see quickly that it is not unique. For one can add to the “tail” of \( f \) a nonzero element of \( H_0^j(K) \) which “begins” at higher order than \( i \), thereby altering the induced element in \( H_0^j(K) \) but not \( \{f\} \) in \( E_\infty^{j,i} \) itself.
What then is the precise relation between this spectral sequence \( \mathcal{E}^i_{jk} \) and the total cohomology \( H^i_\mathcal{F}(K) \)? As we will show now, there is a natural filtration on \( H^i_\mathcal{F}(X) \) whose quotient groups can be identified with \( \mathcal{E}^i_{jk} \).

Let \([f] \) be a nonzero element in \( H^i_\mathcal{F}(K) \). A representative \( f = f_1 + f_{1+1} + f_{1+2} + \cdots \) with \( f_i \in K^{i,j} \) is called reduced, if \( f_i \) represents a nonzero element in \( \mathcal{E}^i_{jk} \).

**Proposition 1.3.** Every nonzero element \([f] \) in \( H^i_\mathcal{F}(K) \) has a reduced representative. Two such reduced representatives of \([f] \) induce the same element in \( \mathcal{E}^i_{jk} \).

**Proof.** Suppose \( f = f_1 + f_{1+1} + f_{1+2} + \cdots \) with \( f_i \in K^{i,j} \) is any representative of \([f] \), then by Theorem 1.1 \( f_i \) represents an element of \( \mathcal{E}^i_{jk} \). Suppose \([f] \) is zero in \( \mathcal{E}^i_{jk} \). Then by Theorem 1.1 there exists an extension \( \text{ext}_g = g + g_1 + \cdots + g_r \) for some \( r \) such that \( D(\text{ext}_g) = 0 \) and \( D(\text{ext}_g) \), \( = f_i \). Then

\[
\tilde{f} = f - D(\text{ext}_g) = (f_{1+1} + D(\text{ext}_g)_{1+1}) + (f_{1+2} + D(\text{ext}_g)_{1+2}) + \cdots
\]

is a new representative of \([f] \). Suppose \([f_{1+1}] \) is zero in \( \mathcal{E}^i_{jk} \). Then there exists an extension \( \text{ext}_h = h_1 + h_2 + \cdots + h_r \) for some \( r \) such that \( D(\text{ext}_h) \cdot = 0 \) and \( D(\text{ext}_h)_{1+1} = f_{1+1} \). As before, we define a new representative \( \tilde{f} = f - D(\text{ext}_g) - (\text{ext}_h) \) whose lowest nonzero term lies in \( K^{r+1,j} \) and so on. If these new representatives always induce the zero element we can define a sequence of extensions \( \text{ext}_g \) such that \( D(\text{ext}_g) \) agrees with \( f \) up to order at least \( r \). As in the previous proof, this sequence stabilizes once we are in the stable range. It is clear that the limit \( \lim_{i \to \infty} \text{ext}_g \) is a cohomology of \( f \) and \( f \) would be zero in \( H^i_\mathcal{F}(K) \).

Hence there must be a representative of \( f = f_1 + f_{1+1} + \cdots \) such that \([f_i] \) is not zero in \( \mathcal{E}^i_{jk} \) for some \( i \).
This is unique by the following reason. Suppose
\[ f = f_1 + f_{i+1} + f_{i+2} + \cdots \]
and
\[ f' = f_1' + f_{i+1} + f_{i+2} + \cdots \]
are equivalent in \( H^i(K) \). Then \( f - f' = Dg \) for some \( g \in K^{i-1} \). Now, if \( i < k \) then \( f_i = (Dg)_1 \) and \([f_i] = 0 \) in \( E_{i+1}^j \), if \( i = k \) then \( f_i - f'_i = (Dg)_i \) and \([f_i] = [f'_i] \) in \( E_{i+1}^j \).

We will call \( i \) the weight of \([f] \) (it can not be larger than the largest \( i \) for which \( E_{i+1}^j \) is unequal to zero). This notion of weight induces a finite filtration on \( H^i(K) \).

Let \( F^i \) be the subspace of elements of \( H^i(K) \) of weight \( i \) or higher. We have then that
\[ H^i(K) / F^i = F^i_1 \uplus F^i_2 \uplus F^i_3 \uplus \cdots \]

(We will denote by \( GH^i \) the associated graded group \( \sum F^i_j / P^i_j \).) We claim that the quotient spaces \( F^i / P^i_1 \) are naturally isomorphic to \( E_{i+1}^j \). Extensions \( \text{ext}_{i+1}^j \) of elements in \( E_{i+1}^j \) obtained via Theorem 1.1 amount to a splitting of the sequence
\[ 0 \rightarrow F^i_{i+1} \rightarrow F^i_i \rightarrow E_{i+1}^j \rightarrow 0 \]

Theorem 1.4. Let \( K = \bigoplus_{i,j} K^{i,j} \) be a doubly graded complex with differential operator \( D : K^i \rightarrow K^{i+1} \). Suppose \( D \) is bounded from the left and induces a bounded spectral sequence \( (Z_i : E_1^{i,j} \rightarrow E_1^{i,j+1}) \). If \( f_i \) in \( K^{i,j} \) denotes the first term of a representative of an element \([f] \in F^i_i \), then the map
\[ \tau : F^i_i \rightarrow E_{i+1}^j \]
\[ [f] \rightarrow [f_i] \]
is an isomorphism of vector spaces.
Proof. Proposition 1.3 shows that $\pi$ is well-defined. If $f = f_1 + f_{1+1} + f_{1+2} + \cdots$, and $f' = f_1' + f_{1+1}' + f_{1+2}' + \cdots$ are representatives for $[f]$ and $[f']$ respectively, then $(f_1 + f'_1) + (f_{1+1} + f_{1+1}') + (f_{1+2} + f_{1+2}') + \cdots$ is a representative for $[f + f']$, hence $\pi$ is clearly a homomorphism. If $f_1$ and $f'_1$ are equal in $\mathcal{E}_0$, then $f$ and $f'$ differ by an element in $F_{1+1}$, so $\pi$ is injective. It is also surjective by Proposition 1.2.

In the following chapter we will apply this spectral sequence and we will see how Spencer's cohomology groups completely determine the first leaf of the sequence. Close examination of the subsequent leaves will show a quite beautiful "web" of increasingly complicated relations on Spencer's groups which after a finite number of steps determines the positive cohomology groups.

2. The Spencer cohomology

A transitive Lie algebra is a filtered Lie algebra $L = L_{-1} \supset L_0 \supset L_1 \supset L_2 \supset \cdots$ with $[L_i, L_j] \subset L_{i+j}$ and $\dim L_i/L_{i+1} < \infty$ that satisfies the completeness condition: whenever $(x_i)$ is a sequence in $L$ such that $x_i - x_{i-1} \in L_i$ for each $i \geq 0$, there is an $x \in L$ such that $x - x_i \in L_i$ for each $i \geq 0$; and the transitivity condition

$$\text{if } x \in L_i \ (i \geq 0) \text{ and } [L_1, x] \subset L_0, \text{ then } x \in L_{i}$$

A graded transitive Lie algebra is a Lie algebra $G = \bigoplus_{i=0}^{\infty} G_i$ with $[G_i, G_j] \subset G_{i+j}$ and $\dim G_i < \infty$ that satisfies the transitivity condition

$$\text{if } x \in G_i \ (i \geq 0) \text{ and } [G_{i+1}, x] = 0, \text{ then } x = 0$$

Every graded transitive Lie algebra is a transitive Lie algebra in a natural manner. Conversely, every transitive Lie algebra induces an "associated graded transitive Lie algebra".
Define

\[ C^{(j)} = \text{Hom}(\Lambda^j G_{-1}, G_{-1}) \]

\[ = \{ f : G_{-1} \times \cdots \times G_{-1} \to G_{-1} \mid f \text{ is multilinear and skewsymmetric} \} \]

\[ \text{times} \]

and the differential operator \( \partial : C^{(j)} \to C^{(j+1)} \) by

\[ (\partial f)(x_1, \ldots, x_{j+1}) = \sum_{k=1}^{j+1} (-1)^{k+1} [f(x_1, \ldots, \hat{x}_k, \ldots, x_{j+1}), x_k]. \]

Then \( \partial^2 = 0 \). The resulting cohomology groups \( H^{(j)}(G) \), for \( i, j \geq 0 \), are called the \textit{Spencer cohomology groups} of the graded algebra \( G \).

Singer and Sternberg [10] proved the important fact that these groups are \textit{bounded}, i.e., for every \( j \) and for sufficiently large \( i \), \( H^{(j)}_{i+1} = 0 \).

It is part of the definition of a graded transitive Lie algebra that \( G_0 \) is a subalgebra acting faithfully on \( G_{-1} \). Let \( A \in G_0 \). Define a map \( f \to f^A \) from \( C^{(j)} \) to itself by

\[ f^A(x_1, \ldots, x_j) = -[A, f(x_1, \ldots, x_j)] + \sum_{k=1}^{j} f(x_1, \ldots, [A, x_k], \ldots, x_j). \]

Since \( (\partial f)^A = \partial (f^A) \), this map induces an action of \( G_0 \) on \( H^{(j)}(G) \).

An element \( f \in H^{(j)}(G) \) is called \textit{invariant} if \( f^A = 0 \) for all \( A \in G_0 \). We will see later that invariance is the first order condition on elements in \( H^{(j)}(G) \) and \( H^{(j)}_{i+1}(G) \) to to define an element in the "positive cohomology" of \( G \).

Given a basis of \( G \) and its structure constants it is easy to calculate these cohomology groups and to show for instance that if \( H^{(j)}_{i+1}(G) = H^{(j)}_{i+1}(G) = 0 \) for all \( i \), then \( G \) is \textit{rigid} or \textit{flat}, i.e., there is no filtered Lie algebra with \( G \) as its associated
graded Lie algebra, other than $G$ itself. Several authors [3,5,7,8] have given other
criteria for the rigidity of $G$ as well as characterized all inequivalent deformations
of $G$ in terms of Spencer cohomology groups. However these theorems all had
to place severe restrictions on higher groups. For if not almost all of them are
zero, certain interrelations between them come into play which become quickly
very complex. We will see that these interrelations are given by a spectral sequence
and the rigidity criteria can be explained as degenerations of that sequence and
thereafter generalized.

3. Positive cohomology.

The Spencer cohomology is peculiar to graded transitive Lie algebras. The
following cohomology, however, can be defined for any graded Lie algebra $G$.

Consider the vector space $K^j = \text{Hom}(\wedge^j G, G)$. We say that $f \in K^j$ has de-
gree $k$, if for any tuple $(e_1, \ldots, e_j)$ with $e_i \in G_{i_1}, \ldots, e_j \in G_{i_j}, f(e_1, \ldots, e_j) \in
G_{i_1+\cdots+i_j+k}$.

Let $K^{j,k} = \text{Hom}_G(\wedge^j G, G)$ and define the differential operator $D : K^{j,k} \rightarrow
K^{j+1,k}$ by

$$D(f(e_1, \ldots, e_{j+1})) = \sum_{i=1}^{j+1} (-1)^i [e_i, f(e_1, \ldots, \hat{e}_i, \ldots, e_{j+1})]
+ \sum_{i<j} (-1)^{i+j} f([e_i, e_j], e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{j+1}).$$

Then $D^2 = 0$. We will call the resulting cohomology groups $H^{j,k}_D(G)$, for $j \geq 0$
and $k \geq 1$ the positive cohomology groups of the graded algebra $G$. These groups,
in particular $H^{j,1}_D$ and $H^{j,2}_D$, are the key to classifying the set $\mathcal{M}$ of all equivalence
classes of transitive Lie algebras with associated graded algebra $G$. (An equivalence
is a Lie algebra isomorphism, which preserves the underlying associated graded Lie algebra.) For example, it is routine to show that every equivalence class $L$ other than $G$ induces a unique non-zero element in $H^k_{\mathfrak{g}}(G)$ for some $k$, and that $G$ is rigid, if $H^k_{\mathfrak{g}}(G) = 0$. Moreover, Rim [9] showed that the set \( \{ a \in H^2_{\mathfrak{g}} \mid a^2 = 0 \text{ in } H^{2,2}_{\mathfrak{g}} \} \) characterises $M$, if $H^k_{\mathfrak{g}} = 0$ for $k \geq 2$ and $H^k_{\mathfrak{g}} = 0$ for $k \geq 3$. Again using spectral sequences and Theorem 1.1 this result has been refined in [12] for situations where the cohomology groups do not vanish.

Remark. If one compares the definitions of $\theta$ and $D$, it is apparent that there must be some relation between their cohomology groups. In fact, we will see that the $i$-gradation of $K^i = \text{Hom}(\mathcal{A}G, G)$, namely $K^i = \text{Hom}(\mathcal{A}G, G_{i-1})$, induces a spectral sequence whose first leaf is entirely determined by Spencer cohomology and whose limit is the positive cohomology. Said in another way, Spencer's groups are the "atoms" out of which positive cohomology groups are made. This fact will give us a way to compute positive cohomology groups and to translate theorems involving positive cohomology groups into theorems involving Spencer's groups, which are favored by geometers for they exhibit certain geometric characteristic classes, such as curvature.

4. The first leaf.

The objective of this section is to examine the first leaf of the spectral sequence on the main complex $K = \prod_{n=1}^{\infty} K^{i, n}$, where $K^{i, n} = \text{Hom}(\mathcal{A}G, G)$, and to show that the sequence is bounded.

Let $G = \prod_{i=1}^{\infty} G_i$ be a graded transitive Lie algebra. Then there exists yet another grading on $K$, namely $K^i = \text{Hom}(\mathcal{A}G, G_{i-1})$. Since $[G_{-1}, G_0] \subset G_{-1}$,
this grading forces the differential operator $D$ given by

$$D(f(x_1, \ldots, x_{j+1})) = \sum_{i=1}^{j+1} (-1)^i [x_i \cdot f(x_1, \ldots, \hat{x_i}, \ldots, x_{j+1})]$$

$$+ \sum_{i=1}^{j+1} \sum_{\ell \neq i} (-1)^{i+\ell+1} f([x_i, x_{\ell}], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_\ell}, \ldots, x_{j+1})$$

to break up into a sequence of maps $D = d_{-1} + d_0 + d_1 + d_2 + \cdots$, where $d_i : K^{ij} \to K^{i+1, j+1}$.

In other words, $D$ is bounded from the left with shift $-1$ with respect to the $i$-grading and hence satisfies Theorem 1.1.

Let us examine the first leaf of the spectral sequence

$$E_{1}^{ij} := \text{H}_{E_{1}}^{ij}(K).$$

What does the operator $d_{-1}$ look like for small $j$? If $f \in K^{ij}$ then we have to pick out the component of $Df$ in $K^{i-1, j+1}$. We treat each $j$ separately:

(j=0) \quad d_{-1} : K^{i0} = G_{-1} \to K^{i-1, 1};

$$d_{-1} f(x) = \begin{cases} [f(x)] & \text{if } x \in G_{-1} \\ 0 & \text{if } x \notin G_{-1} \end{cases}$$

(j=1) \quad d_{-1} : K^{i1} \to K^{i-1, 2};

$$d_{-1} f(x, y) = \begin{cases} -[f(x), y] + [f(y), x] & \text{if } x, y \in G_{-1} \\ + [f(y), x] & \text{if } x \in G_{-1}, y \notin G_{-1} \\ 0 & \text{if } x, y \notin G_{-1} \end{cases}$$
\[(j=2) \quad d_{-1} : K^{i,j} \rightarrow K^{i+1,j} : \]
\[d_{-1} f(x, y, z) = \begin{cases} 
[f(x, y, z)] - [f(x, z, y)] + [f(y, z, x)] & \text{if } x, y, z \in G_{-1} \\
-[f(x, z, y)] + [f(y, z, x)] & \text{if } x, y, z \in G_{-1}, x \notin G_{-1} \\
+f(y, z, x) & \text{if } x, y, z \notin G_{-1} \\
0 & \text{if } x, y, z \notin G_{-1}
\end{cases} \]

\[(j=3) \quad d_{-1} : K^{i,j} \rightarrow K^{i-1,j} : \]
\[d_{-1} f(w, x, y, z) = \begin{cases} 
-f(w, x, y, z) + f(w, x, y) + f(w, y, x) & \text{if } w, x, y \in G_{-1} \\
+f(w, x, y) - f[w, x, y] + f(w, y, x) & \text{if } w, x, y, z \in G_{-1} \\
-f[w, x, y] + f[w, y, x] & \text{if } w, x, y, z \in G_{-1}, x \notin G_{-1} \\
+f[w, y, x] & \text{if } w \in G_{-1}, x, y \notin G_{-1} \\
0 & \text{if } w, x, y \notin G_{-1}
\end{cases} \]

Clearly, \(H_{-1}^{j} = 0\), since \(f \in \ker d_{-1}\) implies \(f = 0\) by the transitivity condition of \(G\). In general, this condition and the last non-zero case in each bracket implies that \(d_{-1}\) is injective on all elements of \(K^{i,j}\) which are non-zero on \(\Delta \Gamma_{p}\). (We will use \(I_{p}\) as an abbreviation of \(\prod I_{p} G_{i}\))

That implies for instance for \(j = 1\)

**Proposition 4.1.**

\[E_{-1}^{1} = H_{-1}^{1}(G) \quad \text{for } i \geq 1.\]

In other words, at dimension one of the first leaf all cohomological elements are Spencer elements.

At dimension \(j = 2\), the operator \(d_{-1}\) again agrees with \(\theta\) on the subcomplex \(C^{i,2} = \text{Hom}(\Delta \Gamma_{G_{i-2}, G_{i-1}})\) and is injective on all elements in \(\text{Hom}(\Delta \Gamma_{G_{i}, G_{i-1}})\). On the complement in \(K^{i-1,2}\) the induced cohomology groups for \(0 \leq p < i\) are

\[H_{-1}^{p} \cong \frac{\{ f \in \text{Hom}(G_{p} \wedge G_{i-1}, G_{i-2}) \mid [f(x, z, y)] - [f(x, y, z)] = 0 \}}{\{ f \in \text{Hom}(G_{p} \wedge G_{i-1}, G_{i-2}) \mid f(x, z, y) = [p(x, z)] \text{ some } \gamma : G_{p} \rightarrow G_{i-1} \}}\]

where \(x, y \in G_{-1}, A \in G_{p}\). In other words, \(H_{-1}^{p} = G_{p} \otimes H_{-1}^{1}\) for \(1 \leq p < i\). It should be mentioned again, that the restriction on \(p\) comes from the fact that the underlying complex \(K\) consists of elements of positive degree only. Summing it up, we have
Proposition 4.2.

$$E_{i-1}^2 = H^{i-1}_{p+q}(G) \oplus \sum_{p+q = i-1} \frac{G_p \otimes H^{i-1}_q}{H^{i-1}_p} \text{ for } i \geq 0.$$ 

At dimension $j = 3$, the complex $K^{i,j}$ should be divided into four subcomplexes. On $C^{i,j}$ the operator $d_{i,j}$ agrees with $\partial$ as before; on $\text{Hom}(\Lambda^i L_q, G_{i-1})$ it is one-to-one. On $\text{Hom}(G_p \wedge G_{-1} \wedge G_{-1}, G_{i-1})$, where $p \geq 0$, it induces the groups $G_p \otimes H^{i-1}_{i-1}$.

$$H^{i-1}_p = \left\{ f \in G_p \otimes C^{i-1} \mid \left[ f(A, x, y, z) - [f(A, y, z), x] + [f(A, y, z), x] = 0 \right] \right\}$$

$$H^{i-1}_{p+q} = \left\{ f \in (G_p \wedge G_q) \otimes C^{i-1} \mid \left[ f(A, B, x, y) - [f(A, B, x), y] = 0 \right] \right\} \text{ for } p + q < i,$$

where $A \in G_p$, $x, y \in G_{-1}$. And on $\text{Hom}(G_p \wedge G_q \wedge G_{-1}, G_{i-1})$, where $p, q \geq 0$ and $p + q < i$, it induces the groups $H^{i-1}_{p+q}$.

Here is a diagram of the first three dimensions of $E_{-1}$:

$$
\begin{array}{ccccccc}
R^{i-1} & H^{i-1}_{p+q} & H^{i-1}_{p+q+1} & \cdots & H^{i-1}_{p+q+r-1} & H^{i-1}_{p+q+r} & \cdots \\
H^{i-1}_{p+q+r} & H^{i-1}_{p+q+r+1} & \cdots & H^{i-1}_{p+q+r+s-1} & H^{i-1}_{p+q+r+s} & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

Proposition 4.3.

$$E_{i-1}^2 = H^{i-1}_{p+q}(G) \oplus \left( \sum_{p+q = i-1} \frac{G_p \otimes H^{i-1}_q}{H^{i-1}_p} \right) \oplus \left( \sum_{p+q = i-1} \frac{G_p \wedge G_q \otimes H^{i-1}_q}{H^{i-1}_p} \right) \text{ for } i \geq 0.$$
We can now see for $0 \leq j \leq 3$ (and it is true and easy to see for all $j$) that each $E_{j+1}^d$ is finite dimensional since $G_j$ and $H^d_j(G)$ are all finite dimensional and that the $E_{j+1}^d$ are bounded for each $j$ since the $H^d_j(G)$ are bounded by Singer and Sternberg's Theorem. Applying Theorem 4.4 this proves the important fact

**Theorem 4.4.** Let $G$ be a graded Lie algebra. There exists a bounded spectral sequence $(E_{r-1}^{i,j})_{r \geq 1}$ with $d_r : E_{r-1}^{i,j} \rightarrow E_{r+1}^{i+1,j+1}$ converging to the positive cohomology of $G$ with first term

$$E_r^{i,j} = \bigoplus_{n=0}^{i-1} H^d_{i-j-n} \otimes (\bigotimes_{m+1} \cdots \bigotimes_{m+n} G_{m})$$

and with limit

$$\sum_r E_r^{i,j} = GH^d_i$$

Since $E_r^{i,j}$ can only be smaller than $E_{r+1}^{i,j}$, it follows from Theorem 4.4 that

**Corollary 4.5.** Positive cohomology groups $H^d_j(G)$ are finite dimensional for all $j$ and are zero for $j = 0$. 

or in shorthand

\[
\begin{align*}
H^0_0 + I^0_0 & \quad H^1_0 + I^1_0 + H^1_0 + J^1_0 + \cdots + H^1_0 + \sum_{\mu=1}^1 I^1_\mu + \sum_{\nu=3}^1 J^1_\nu \\
H^2_0 & \quad H^2_0 + I^2_0 + \cdots + H^2_0 + \sum_{\mu=3}^2 I^2_\mu \\
H^3_0 & \quad \cdots \\
H^4_0 & \quad \cdots 
\end{align*}
\]
Corollary 4.6. If for some \( r \), \( E_i^{2r} = 0 \) for all \( i \geq 0 \), then \( G \) is rigid.

Proof. By Theorem 4.4 it follows from the hypotheses that \( H^0_\mathcal{B}(G) = 0 \). Therefore \( G \) is rigid.

Corollary 4.7. (The first order criterion) If \( H^0_\mathcal{B} = 0 \) for \( i \geq 1 \) and \( H^0_{\mathcal{B}_k} = 0 \) for \( i \geq 0 \), then \( G \) is rigid.

Proof. By Proposition 4.2, \( E^0_i = 0 \) for all \( i \).

Before concluding this section, let us return once more to the diagram that accompanied Proposition 4.3. The differential operator \( D = d_{-1} + d_0 + d_1 + d_2 + \cdots \) preserves the degree of maps and so do all the \( d_r \). Hence we can split up the diagram into more parts. In fact, it is really a three-dimensional diagram with coordinates \( i, j \) and degree \( k \). That will be very helpful in order to see the domain and range of the various maps \( d_r \) and the induced differential operators \( \delta_r \) and to understand the intricate "web" they draw between the various Spencer groups.

For instance, consider the term \( H^1_{\mathcal{B}} + I^1_{\mathcal{B}} + I^2_{\mathcal{B}} + I^3_{\mathcal{B}} + I^4_{\mathcal{B}} \). Elements in \( H^1_{\mathcal{B}} \) map \( G_{-1} \times G_{-2} \times G_{-3} \) into \( G_0 \), so have degree 3. Elements in \( I^1_{\mathcal{B}} \) map \( G_{-3} \times G_{-1} \times G_0 \) into \( G_0 \), so have degree 2. Elements in \( I^2_{\mathcal{B}} \) map \( G_{-1} \times G_{-1} \times G_1 \) into \( G_0 \), so have degree 1. Elements in \( I^3_{\mathcal{B}} \) map \( G_{-1} \times G_0 \times G_0 \) into \( G_0 \), so have also degree 1. We will denote from now on by \( E^1_{\mathcal{B}} \) the cohomological elements with range \( i \), dimension \( j \) and degree \( k \) of the leaf \( r \). So, for example, \( E^3_{\mathcal{B}} \) denotes \( I^3_{\mathcal{B}} + I^4_{\mathcal{B}} \).

The fact that \( D \) preserves the degree \( k \) then gives us actually a refinement of the above Theorem:
Theorem 4.8. Let $G$ be a graded Lie algebra. There exists a bounded spectral sequence $\{E_2, d_r\}_{r \geq 1}$ with $d_r : \mathcal{E}^{p,q}_{r-1} \to \mathcal{E}^{p+r,q}_{r-1}$ converging to the positive cohomology of $G$ with first term

$$E_2^{p,q} = \sum_{r=1}^{\infty} H^p_{d_r} \otimes \bigoplus_{q \geq 0} G_{q+r} \wedge \cdots \wedge G_{q+r}$$

and with limit

$$\lim_{r \to \infty} E_r^{p,q} = \mathcal{H}^p_{d_r} (G).$$

This theorem is summarized in Table 1.

5. The second leaf

In section 1 it was shown that if $D = d_{-1} + d_0 + d_1 + \cdots$ is a differential operator on $K$, then $d_0$ acts as a differential operator on $H^1_{d_0} (K)$, in this chapter we will examine the structure of the resulting groups

$$\mathcal{H}^1_{d_0} (K) := H^1_{d_0} (H_{d_0} (K)).$$

In dimension one, $D : K^1 \to K^3$ is given by

$$Df(x,y) = -f(x,y) + f(y,x) + f(x,y).$$

If $f \in K^4$, then $d_0$ acts as a differential operator on $K^4$, but it acts as a differential operator on $K^4$. Therefore $d_0 : K^4 \to K^4$ amounts to:

$$d_0 f(x,y) = \begin{cases}
-f(x,y) + f(y,x) & \text{if } x, y \in G_{-1} \\
-f(x,y) + f(y,x) + f(x,y) & \text{if } x \in G_{-1}, y \in G_0 \\
-f(x,y) + f(y,x) + f(x,y) & \text{if } x \in G_0, y \in G_0 \\
-f(x,y) + f(y,x) & \text{if } x, y \in G_0, k \geq 1
\end{cases}$$
<table>
<thead>
<tr>
<th>H_j</th>
<th>H_j^2</th>
<th>\ldots</th>
<th>H_j^{n_j}</th>
</tr>
</thead>
<tbody>
<tr>
<td>H_{ij}</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>H_{ij}^3</td>
<td>\ldots</td>
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<td>\ldots</td>
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</tr>
<tr>
<td>H_{ij}^2 \cdot H_{ij}</td>
<td>H_{ij}^3 + H_{ij}^2</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>H_{ij}^3 \cdot H_{ij}^2</td>
<td>H_{ij}^3 \cdot H_{ij}^2 + H_{ij}</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

**Table 1**

- \( H_{ij} \)
- \( H_{ij}^2 \)
- \( H_{ij}^3 \)

\[ H_{ij}^2 \cdot H_{ij}^3 \]

\[ H_{ij}^2 \cdot H_{ij}^3 + H_{ij}^2 \cdot H_{ij}^3 \]

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\[ H_{ij}^3 \cdot H_{ij}^2 + H_{ij}^3 + H_{ij}^2 + H_{ij} \]
On the level of cohomology the only non zero map is

\[ d_0 : H^1_S(G) \to H^2_S = G_S \oplus H^1_S \]

\[ (d_0 f)(z,y) = -f(z,y) + f([x,y]) \]

where \( x \in G_{-1}, y \in G \).

So the cohomological elements of order two, which we will denote by \( H^2 \) are precisely the invariant elements in \( H^2_S \). Hence

**Proposition 5.1.** Level one of the second leaf \( H^1_S = H^1_S(H^1_S) = H^2 \) consists of all invariant elements in \( H^2_S \).

In dimension two, \( D : K^2 \to K^3 \) is given by \( Df(x,y,z) = [f(x,y), z] - [f(x,z), y] + [f(y,z), x] + f([x,y], z) - f([x,z], y) + f([y,z], x) \).

Again, if \( f \in K^{i+1} \), then \( d_0 \) picks out the component of \( Df \) in \( K^{i+1} \).

Broken up into its various cases, \( d_0 : K^{i+2} \to K^{i+3} \) then looks like

\[
\begin{align*}
\text{for } & f(x,y) z \in \begin{cases}
+ & [f(x,y),z] \\
- & f([x,z],y) + f([y,z],x)
\end{cases} \quad \text{if } \begin{cases}
\{ x,y \in G_{-1} \} \\
\{ z \in G_0 \}
\end{cases} \\
& f([x,y],z) - f([x,z],y) + f([y,z],x) \quad \text{if } \begin{cases}
\{ z \in G_{-1} \} \\
\{ x \in G_0 \}
\end{cases}
\end{align*}
\]
In cohomology only the following five maps are non-zero (for clarity let $x_k$ denote that $x \in G_k$):

First,  
\[
0 : H^{2k}_G \to H^{2k}_G \equiv G_k \otimes H^{2k}_G
\]
\[
d_0 f(x, y, z) = f(x, y, z) - f([x, y], z) + f([y, z], x) = f(x, y, z) - f([x, y], z) + f([y, z], x).
\]

Hence

Lemma 5.2. $H^{2k}_G := H_{G_k}(H^{2k}_G)$ consists of all invariant elements in $H^{2k}_G$.

Then,  
\[
d_0 : H^{2k}_G \to H^{2k+1}_G \equiv G_k \otimes H^{2k+1}_G, \text{ where } k \geq 0,
\]
\[
d_0 f(x, y, z) = -f([x, y], z) + f([y, z], x), \text{ and } d_0 H^{2k}_G \to H^{2k+1}_G = (G_k \otimes G_k) \otimes H^{2k+1}_G,
\]
\[
d_0 f(x, y, z) = f(x, y, z) - f([x, y], z) + f([y, z], x) - f([x, y], z) + f([y, z], x).
\]

Therefore, $f(x, y) - f(x, y) - f(y, z)$ if we think of $f$ as $G_k \to H^{2k}_G$. Hence

Lemma 5.3. $H^{2k}_G := H_{G_k}(H^{2k}_G)$ is the set

$$
\left\{ f : G_k \to H^{2k}_G \mid \begin{array}{ll}
(1) & f(x, y, z) - f([x, y], z) + f([y, z], x) \\
(2) & - f([x, y], z) + f([y, z], x) \in H^{2k}_G
\end{array} \right\}
$$

Finally,  
\[
d_0 : H^{2k}_G \to H^{2k+1}_G = (G_k \otimes G_k) \otimes H^{2k+1}_G, \text{ where } k \geq 1,
\]
\[
d_0 f(x, y, z) = -f([x, y], z) + f([y, z], x) + f([y, z], x),
\]
\[
d_0 : H^{2k}_G \to H^{2k+1}_G = (G_k \otimes G_k) \otimes H^{2k+1}_G, \text{ where } k \geq 0 \text{ and } p + q = k,
\]
\[
d_0 f(x, y, z) = f([y, z], x).
\]

Hence

Lemma 5.4. For $0 < k < 0$, $H^{2k}_G := H_{G_k}(H^{2k}_G)$ is the set

$$
\left\{ f : G_k \to H^{2k}_G \mid \begin{array}{ll}
(1) & f(x, y, z) - f([x, y], z) + f([y, z], x) = 0 \\
(2) & - f([x, y], z) + f([y, z], x) = 0 \in H^{2k+1}_G
\end{array} \right\}
$$

$$
\left\{ f : G_k \to H^{2k}_G \mid f([y, z], x) = 0 \right\}
$$
Lemma 5.2 through 5.4 describe explicitly the crucial dimensions of the second leaf of our spectral sequence and we can formulate the following criterion for rigidity of $G$:

**Proposition 5.5.** (the second order criterion) Let $G$ be a graded Lie algebra. Suppose that

1. $H^{i,2} = 0$ for all $i \geq 0$,
2. $I^{i,0} = 0$ for all $i \geq 1$,
3. $I^{i,0} = 0$ for all $i \geq 2$ and $1 \leq k < i$,

then $G$ is rigid.

**Proof.** Since $I^{i,0} = H_{i,0}^{i,0}(G) = H^{i,0} \oplus I^{i,0} \oplus \sum_{k=1}^{i-1} I^{i,k}$, it follows from

Corollary 4.6. 

**Remark.** This second order criterion corresponds very closely to Koch's Theorem in [6]. Recall

**Theorem 5.6 (Koch).** Let $G$ be a graded Lie algebra. Suppose

1. $H^{i,0}(G)$ has no non-zero invariant elements for $i \geq 0$
2. $G_{k} = H^{i,k} |_{\ker H^{i,k} = \ker H^{i-1,k}}$ for some $k \leq i$.
3. $H^{i,0}(G, H^{i,k}) = 0$ for $1 \leq k < i$.

Then $G$ is rigid.

Upon close examination we find that (5.5) is actually a slightly stronger theorem: hypotheses (1) is the same as (a), (2) implies (b), and (3) implies (c). Said in another way, Koch's Theorem requires slightly more elements to be zero in cohomology than (5.5). It can be shown though that the original proof of (5.6) can easily be extended to prove the somewhat sharper result (5.5).
Generally speaking, it is a strength of this approach via spectral sequences that it is possible to predict certain results which can be proven directly without the use of spectral sequences in a somewhat more lucid way. Furthermore it is considerably easier to see the generalizations to higher orders. For example, it is now routine to calculate the proper "third order criterion". That, however, looks too complex to be of much value here. We rather stress that for any given transitive graded Lie algebra $G$ the "tic-tac-toe scheme" lined out in section 1 makes it possible, beginning with Spencer cohomology groups, to calculate all positive cohomology groups with explicit representatives for each cohomology class.

6. Examples

The Euclidean Lie Algebra

Consider the graded transitive Lie algebra $E(3) := R^3 \oplus O(3)$. Clearly $E(3)$ is generated by

$$G_{-1} : \quad X = \tfrac{1}{2} X_1, \quad Y = \tfrac{1}{2} X_2, \quad Z = \tfrac{1}{2} X_3.$$

$$G_3 : \quad A = x \frac{1}{2} X_1 - y \frac{1}{2} X_2, \quad B = x \frac{1}{2} X_1 - z \frac{1}{2} X_3, \quad C = y \frac{1}{2} X_2 - z \frac{1}{2} X_3.$$

The brackets are

$$[A, X] = -Y, \quad [B, X] = -Z, \quad [C, X] = 0, \quad [A, B] = -C,$$

$$[A, Y] = X, \quad [B, Y] = 0, \quad [C, Y] = -Z, \quad [A, C] = B,$$


All $H^{1,1}_A$ vanish. Also $H^{1,3}_A = 0$, since $\dim \operatorname{Hom}(G_{-1}, G_3) = 9$, which is also equal to $\dim \operatorname{Hom}(G_{-1} \wedge G_{-1}, G_3)$. $H^{1,3}_A$ is six dimensional and a basis is given by the
functions

\[
\begin{cases}
  f_1(X, Y) = A \\
  f_3(X, Z) = B \\
  f_4(Y, Z) = C
\end{cases}
\quad \text{and} \quad
\begin{cases}
  f_2(X, Y) = A \\
  f_3(Y, Z) = B \\
  f_4(X, Z) = C
\end{cases}
\]

\( H^{1,3}_p \) is three dimensional but is irrelevant. All other Spencer groups are clearly zero. So the first leaf \( E_0 \) looks like this:

\[
\begin{pmatrix}
    \ast & 0 & 0 & 0 \\
    0 & G_0 \otimes H^{1,2}_p & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

Which elements in \( H^{1,2}_p \) are invariant? Let us compute the second leaf, which is also the final leaf since \( \delta_i = 0 \). Recall that

\[
f'(x, y) = [f(x, y), x] - f([x, z], y) + f([y, z], x).
\]

One quickly calculates that

\[
\begin{align*}
  f_1^4 &= 0, & f_2^4 &= -f_5, & f_3^4 &= f_4, & f_4^4 &= \frac{1}{2}(f_1 + f_2), \\
  f_1^5 &= f_5, & f_2^5 &= 0, & f_3^5 &= -f_5, & f_4^5 &= f_4, \\
  f_1^6 &= -f_5, & f_2^6 &= f_5, & f_3^6 &= 0, & f_4^6 &= f_4.
\end{align*}
\]
Hence the subspace of invariant elements in $E_f^2$ is one dimensional and generated by the element $\mu = f_1 + f_2 + f_3$. To be explicit, $\mu$ is given by:

\[
\begin{align*}
\mu(X,Y) &= A \\
\mu(X,Z) &= B \\
\mu(Y,Z) &= C
\end{align*}
\]

and the positive cohomology groups of $E(3)$ are

\[
\begin{align*}
H^2_B &= 0 \\
H^3_B &= \lambda \cdot \mu.
\end{align*}
\]

The corresponding deformations of $E(3)$ are of course the filtered Lorentzian Lie algebra $L(3) \supset O(3)$ for $\lambda > 0$, and the spherical algebra $O(4) \supset O(3)$ for $\lambda < 0$.

Koch's Algebras $G_{\lambda,n}$

In [8], Koch examines the algebras

\[
G_{\lambda,n} = \left\{ (f(x) + \lambda P(x)) \frac{\partial}{\partial x} + P(y) \frac{\partial}{\partial y} \mid \deg P \leq 1, \deg f \leq n \right\}
\]

and finds that $G_{\lambda,n}$ satisfy the "second order criterion" (Theorem 5.6), and hence are rigid, for all $n \leq \infty$ and $\lambda$ except for the critical values $\lambda = -1, 0, n + 1$. It is found by direct calculation that for $n < \infty$ and $\lambda = -1, 0$, the graded algebras $G_{\lambda,n}$ are rigid. We will see that this can be explained as a higher order phenomenon.

For $n = \infty$, both values $-1$ and $0$ do correspond to non-graded algebras and we will show for $\lambda = -1$ how to calculate its corresponding infinitesimal deformation in $H^2_B$. 

\[ G_{\ast \ast} \] for \( \lambda = -1 \) and \( n = 2 \)

Consider the algebra \( G = \{ (f(y) - P(x)) \delta y + P(y) \delta x \mid \deg P \leq 1, \deg f \leq 2 \} \).

It is graded and is generated by

\[ \begin{align*}
G_{-1} & : \quad X = \delta y, \quad Y = \delta x, \\
G_0 & : \quad A = y \delta y, \quad B = y \delta y - x \delta x, \\
G_1 & : \quad C = x^2 \delta x.
\end{align*} \]

The brackets are

\[ \begin{align*}
[A, X] & = 0, \quad [B, X] = X, \quad [C, X] = 0, \quad [C, A] = 0, \\
\end{align*} \]

The only non-vanishing cohomology groups are \( H^{1,1}_0, H^{1,2}_0, H^{1,1}_2, \) and \( H^{2,2}_0 \). All are one-dimensional. \( H^{1,1}_0 \) is generated by \( f(X) = -A, f(Y) = B; H^{2,2}_0 \) by \( g(X) = 0, g(Y) = C; H^{2,2}_2 \) by \( h(X, Y) = C; \) and \( H^{3,2}_2 \) by \( c(X, Y) = B \).

Table 2 is a diagram of the first leaf \( (E_0, d_0) \).

However, only \( [c] \in H^{3,2}_2 \) is a cocycle of \( d_0 \) and "survives" until the second leaf.

In fact, \( d_0c = d_{-1}b \) where \( b: G_{-1} \wedge G_0 \to G_1 \) is given by \( h(x, A) = 2C \). Table 3 is a diagram of the second leaf \( (E_1, d_1) \).

Finally, even the last element \( [c] \in H^{3,2}_2 \) does not survive until the third and stable leaf as we will show now. Recall that \( d_1c \) is given by

\[ \begin{align*}
[d_1c] & = d_1c - d_0b, \quad \text{where} \quad d_0c = d_{-1}b
\end{align*} \]

and

\[ \begin{align*}
c: G_{-1} \wedge G_{-1} & \to G_0 \quad \text{and} \quad b: G_{-1} \wedge G_0 \to G_1, \\
c(X, Y) & = B \quad \text{and} \quad h(X, A) = 2C.
\end{align*} \]
Hence, 
\[ \overline{\partial}_1 c(X, Y, C) = [c(X, Y), C] - k([X, C], Y) - k(X, [Y, C]) = C \]
\[ \overline{\partial}_2 c(X, A, B) = [k(X, B), A] - [k(X, A), B] - k([X, A], B) \]
\[ + k(X, [B, A]) - k([A, B], X) \]
\[ = 0 \]
\[ \overline{\partial}_3 c(Y, A, B) = [k(Y, B), A] - [k(Y, A), B] - k([Y, A], B) \]
\[ + k(Y, [B, A]) - k([A, B], Y) \]
\[ = 0. \]
So \( \overline{\partial}_1 c(C) = h \) as maps is \( H^{2,2}_B \). In particular, \([c] \notin E_1^{2,2} \). Hence \( E_1^{2,2} = E_0^{2,2} = 0 \) for all \( i \), and \( G \) is rigid.

\[ G_{n+1} \text{ for } n = -1 \text{ and } n = \infty \]

The algebra \( G = \left\{ f(y) - F(z) \right\} B^P + P(y) B^P \mid \deg P \leq 1, f \in \mathcal{H}[\|y\|] \} \) is closely related to the previous example. However it is infinite dimensional and does not have any obstructions at \( i > 1 \). The cohomological element \( c \in H^{12} \) does survive until \( \infty \) and defines an element in the group \( H_B^{12} \). By following \( c \) through the leaves of the spectral sequence and applying the "tic-tac-toe scheme" of section 1, it is possible to obtain an arbitrarily long extension for \( c \), which in the limit defines a representative for \([c] \in H_B^{12} \).

\( G \) is generated by

\[ G_{-1} : \quad X = \frac{y}{z^2}, \quad Y = \frac{y}{z^2} \]

\[ G_0 : \quad A_0 = y \frac{z}{y}, \quad B = y \frac{z}{y} - z \frac{z}{y} \]

\[ G_i : \quad A_i = \frac{y^i}{z^{i+2}} \frac{z}{y}, \quad i = 1, 2, 3, \ldots. \]
One checks that $H^{1,3}_B$ and $H^{1,3}_H$ are the same as before. But all other groups are zero. Again the only invariant element is $c \in H^{1,3}_B$. So the second leaf is:

We can read off the fact that $[c]$ defines a non-zero element in $H^2_B$ and that by Proposition 1.2 there exists an representative $\text{ext } c = c + b_0 + b_1 + b_2 + \cdots$ such that $D(\text{ext } c) = 0$. So far we have

$$c : G_{-1} \wedge G_0 \rightarrow G_0$$

and

$$b_0 : G_{-1} \wedge G_0 \rightarrow G_1$$

Claim 6.1. There exists a representative for $[c] \in H^2_B$ of the form

$$\text{ext } c = c + b_0 + b_1 + (b_0 + b_1) + (b_0 + b_1 + b_2) + \cdots$$

such that

$$b_n : G_{-1} \wedge G_n \rightarrow G_{n+1}$$

$$b_n(B, A_n) = b_n = A_{n+1}$$

and

$$b_n(A_n, A_{n+1}) = b_n = A_{n+2}$$

In fact,

$$b_n = \begin{cases} \sum \frac{a_{n+3}}{a_{n+2}} & \text{if } n \text{ is even} \\ a_{n+2} \sum \frac{a_{n+3}}{a_{n+2}} \sum \frac{a_k}{k} & \text{if } n \text{ is odd} \end{cases}$$
Proof by induction on \( n \). \( \delta \) is non-zero only for \( (X, Y, A_{n+1}) \) and \( (Y, A_r, A_s) \), where \( r + s = n \). By Theorem 1.1, \( \delta_{n+1} \) is given by

\[
\delta_{n+1}(X, Y, A_{n+1}) = \delta_{n+1}(C) - \delta_{n+1}(A_{n+1}) - \cdots - \delta_{n+1}(A_1) + \sum_{k=1}^{n+1} b_k
\]

\[
= \left( [X, Y], A_{n+1} \right) - b_n [X, Y, A_{n+1}] - b_{n+1} [X, A_{n+1}], Y) + \sum_{k=1}^{n+1} b_k
\]

\[
= [Y, A_{n+1}] - b_n (X, A_r) - b_{n+1}
\]

\[
= (n + 2) \cdot \delta_{n+1} - b_n \cdot A_{n+1} = \delta_{n+1}(X, Y, A_{n+1}).
\]

So we can choose \( b_{n+1}(X, A_{n+1}) = (n + 2 - b_n) \cdot A_{n+1} \). Furthermore

\[
\delta_{n+1}(Y, A_r, A_s) = \delta_{n+1}(C) - \delta_{n+1}(A_{n+1}) - \cdots - \delta_{n+1}(A_1) + \sum_{k=1}^{n+1} b_k
\]

\[
= -b_n \cdot [Y, A_r], A_s) - b_{n+1} \cdot (Y, A_s)] + \sum_{k=1}^{n+1} b_k
\]

\[
= b_{n+1} \cdot A_{n+1} - b_{n+1} \cdot A_{n+1} = \delta_{n+1}(Y, A_r, A_s),
\]

which gives \( b_n = b_{n+1} + b_{n+1} \).

REFERENCES


